

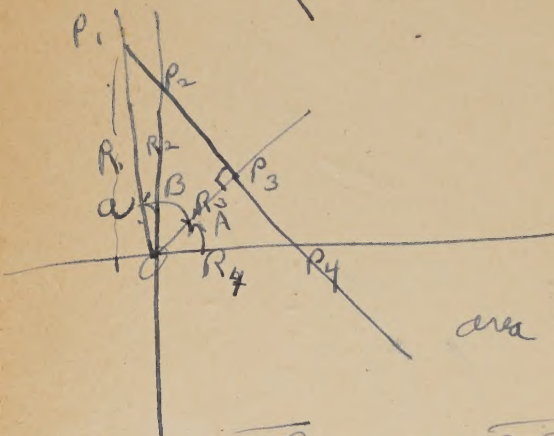
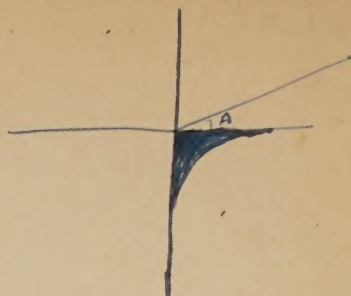
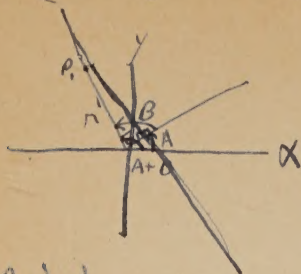
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$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$



$$\text{area } P_1 O P_4 = \text{area } P_1 O P_3 + \text{area } P_3 O P_4$$

$$R_3 \cdot \overline{P_1 P_3} \quad R_3 \quad \overline{P_3 P_4}$$

$$= R_3 R_1 \sin B + R_3 R_4 \sin A$$

$$a R_4$$

$$R_4 \{ r, \sin(A+B) \} =$$

$$\tan A + B = \frac{\sin(A+B)}{\cos(A+B)}$$

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COMPLETE TRIGONOMETRY

BY

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INSTITUTE OF TECHNOLOGY



BOSTON, U.S.A.

D. C. HEATH & CO., PUBLISHERS

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PREFACE.

THE present volume is a revision of the author's *Essentials of Trigonometry*.

In preparing the new edition, many improvements have been effected; the attention of teachers is specially invited to the following features of the work:

1. The proofs of the functions of 120° , 135° , 150° , etc.; § 27.

2. The proofs of the functions of $(-A)$, and $(90^\circ + A)$, in terms of those of A ; §§ 28, 29.

3. The method of solution employed in the examples of §§ 33 and 34.

4. The general demonstration of the formulæ

$$\tan x = \frac{\sin x}{\cos x}, \text{ and } \sin^2 x + \cos^2 x = 1,$$

in §§ 36 and 38; the four cases being considered together.

5. The general demonstrations of the formulæ

$$\cot x = \frac{\cos x}{\sin x}, \sec^2 x = 1 + \tan^2 x, \text{ and } \csc^2 x = 1 + \cot^2 x$$

in §§ 37 and 39.

6. The proofs of the formulæ for $\sin(x+y)$ and $\cos(x+y)$, when x and y are acute; the two cases when $x+y$ is acute or obtuse being considered together; § 41.

7. The proofs of the formulæ for $\tan \frac{1}{2}x$ and $\cot \frac{1}{2}x$; § 48.

8. The illustrative examples in § 49.

9. The solution of right triangles by Natural Functions; see Ex. 1, page 66.

The new work contains a great many more examples than the old; they have been selected with great care, and most of them are new. It is not expected that every class will solve all the examples; they are sufficiently numerous to furnish a variety in successive years.

Attention is specially invited to the sets in §§ 96, 114, 157, and 160.

In § 112 will be found a set of miscellaneous examples in the solution of plane oblique triangles, and in § 155 a set in spherical oblique.

In the Appendix to the Plane Trigonometry there will be found a discussion of the principles of Parallel, Middle Latitude, and Traverse Sailing, with problems.

The results have been worked out by aid of the author's New Four Place Logarithmic Tables, which contain also Tables of Natural Functions.

WEBSTER WELLS.

MASSACHUSETTS INSTITUTE OF
TECHNOLOGY, 1900.

CONTENTS.

PLANE TRIGONOMETRY.

	PAGE
I. TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES	1
II. TRIGONOMETRIC FUNCTIONS OF ANGLES IN GENERAL	7
III. GENERAL FORMULÆ	26
IV. MISCELLANEOUS THEOREMS	38
V. LOGARITHMS	50
Properties of Logarithms	52
Applications	58
VI. SOLUTION OF RIGHT TRIANGLES	65
Formulæ for the Area of a Right Triangle	72
VII. GENERAL PROPERTIES OF TRIANGLES	75
Formulæ for the Area of an Oblique Triangle	80
VIII. SOLUTION OF OBLIQUE TRIANGLES	82
APPENDIX TO PLANE TRIGONOMETRY	96
Parallel Sailing	96
Middle Latitude Sailing	96 a
Traverse Sailing	96 c

SPHERICAL TRIGONOMETRY.

	PAGE
IX. GEOMETRICAL PRINCIPLES	97
X. RIGHT SPHERICAL TRIANGLES	101
Solution of Right Spherical Triangles	108
XI. OBLIQUE SPHERICAL TRIANGLES	118
General Properties of Spherical Triangles	118
Napier's Analogies	124
Solution of Oblique Spherical Triangles	127
XII. APPLICATIONS	139
FORMULÆ:	
Plane Trigonometry	144
Spherical Trigonometry	147
ANSWERS	
USE OF THE TABLES	

PLANE TRIGONOMETRY.

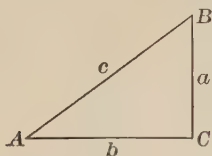


I. TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES.

1. Trigonometry treats of the properties and measurement of angles and triangles.

In *Plane Trigonometry*, we consider *plane* figures only.

2. Definitions of the Trigonometric Functions of Acute Angles.



Let BAC be any acute angle.

From any point in either side, as B , draw line BC perpendicular to AC , forming right triangle ABC .

We then have the following definitions, applicable to either of the acute angles A or B :

In any right triangle,

The **sine** of either acute angle is the ratio of the opposite side to the hypotenuse.

The **cosine** is the ratio of the adjacent side to the hypotenuse.

The **tangent** is the ratio of the opposite side to the adjacent side.

The **cotangent** is the ratio of the adjacent side to the opposite side.

The **secant** is the ratio of the hypotenuse to the adjacent side.

The **cosecant** is the ratio of the hypotenuse to the opposite side.

We also have the following definitions:

The **versed sine** of an angle is 1 minus the cosine of the angle.

The **covered sine** is 1 minus the sine.

The eight ratios defined above are called the *Trigonometric Functions* of the angle:

Representing sides BC , CA , and AB , by a , b , and c , respectively, and employing the usual abbreviations, we have:

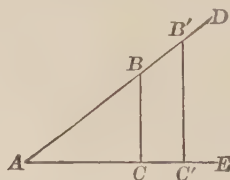
$$\sin A = \frac{a}{c} \quad \tan A = \frac{a}{b} \quad \sec A = \frac{c}{b} \quad \text{vers } A = 1 - \frac{b}{c}$$

$$\cos A = \frac{b}{c} \quad \cot A = \frac{b}{a} \quad \csc A = \frac{c}{a} \quad \text{covers } A = 1 - \frac{a}{c}$$

$$\sin B = \frac{b}{c} \quad \tan B = \frac{b}{a} \quad \sec B = \frac{c}{a} \quad \text{vers } B = 1 - \frac{a}{c}$$

$$\cos B = \frac{a}{c} \quad \cot B = \frac{a}{b} \quad \csc B = \frac{c}{b} \quad \text{covers } B = 1 - \frac{b}{c}$$

3. It is important to observe that the values of the trigonometric functions depend solely on the magnitude of the angle, and are entirely independent of the lengths of the sides of the right triangle which contains it.



For let B and B' be any two points in side AD of angle DAE , and draw lines BC and $B'C'$ perpendicular to AE .

Then, by the definitions of § 2,

$$\sin A = \frac{BC}{AB}, \text{ and } \sin A = \frac{B'C''}{AB'}.$$

But right triangles ABC and $AB'C''$ are similar, since they have angle A common.

Whence, by Geometry,

$$\frac{BC}{AB} = \frac{B'C''}{AB'}.$$

Thus the two values found for $\sin A$ are equal.

The same may be proved true of each of the remaining functions.

4. We have from § 2,

$$\sin A = \cos B.$$

$$\sec A = \csc B.$$

$$\tan A = \cot B.$$

$$\text{vers } A = \text{covers } B.$$

But B is the complement of A .

Hence, *the sine, tangent, secant, and versed sine of any acute angle are, respectively, the cosine, cotangent, cosecant, and coversed sine of the complement of the angle.*

5. From § 2, $\sin A = \frac{a}{c} = \cos B$, and $\cos A = \frac{b}{c} = \sin B$.

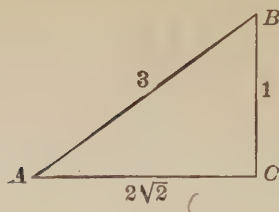
Whence,

$$a = c \sin A = c \cos B, \text{ and } b = c \sin B = c \cos A.$$

That is, *in any right triangle, either side about the right angle is equal to the sine of the opposite angle, or the cosine of the adjacent angle, multiplied by the hypotenuse.*

6. To find the Values of the Other Seven Functions of an Acute Angle, when the Value of Any One is Given.

1. Given $\csc A = 3$; find the values of the remaining functions of A .



We may write the equation $\csc A = \frac{3}{1}$.

Since the cosecant is the hypotenuse divided by the opposite side, we may regard A as one of the acute angles of right triangle ABC , in which hypotenuse $AB = 3$, and opposite side $BC = 1$.

By Geometry, $AC = \sqrt{AB^2 - BC^2} = \sqrt{9 - 1} = \sqrt{8} = 2\sqrt{2}$.

Then by the definitions of § 2,

$$\sin A = \frac{1}{3}.$$

$$\tan A = \frac{1}{2\sqrt{2}}.$$

$$\sec A = \frac{3}{2\sqrt{2}}.$$

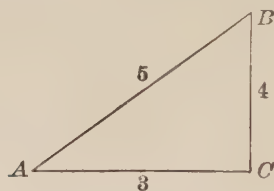
$$\cos A = \frac{2\sqrt{2}}{3}.$$

$$\cot A = 2\sqrt{2}.$$

$$\text{vers } A = 1 - \frac{2\sqrt{2}}{3}.$$

$$\text{covers } A = 1 - \frac{1}{3} = \frac{2}{3}.$$

2. Given $\text{vers } A = \frac{2}{5}$; find the value of $\cot A$.



Since $\text{vers } A = 1 - \cos A$, we have $\cos A = 1 - \frac{2}{5} = \frac{3}{5}$.

Then, in right triangle ABC , we take adjacent side $AC = 3$, and hypotenuse $AB = 5$.

Whence, $BC = \sqrt{AB^2 - AC^2} = \sqrt{25 - 9} = \sqrt{16} = 4$.

Then, by definition, $\cot A = \frac{3}{4}$.

EXAMPLES.

In each of the following, find the values of the remaining functions :

$$3. \quad \sin A = \frac{3}{5}. \quad 6. \quad \csc A = 7. \quad 9. \quad \sec A = x.$$

$$4. \quad \text{vers } A = \frac{8}{13}. \quad 7. \quad \cos A = \frac{3\sqrt{3}}{14}. \quad 10. \quad \tan A = \frac{1}{x}.$$

$$5. \quad \cot A = \frac{7}{24}. \quad 8. \quad \text{covers } A = \frac{2}{17}. \quad 11. \quad \sin A = \frac{a}{b}.$$

$$12. \quad \text{Given } \cot A = \frac{3}{2}; \text{ find } \sin A.$$

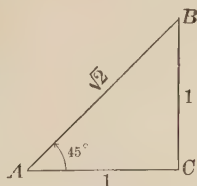
$$13. \quad \text{Given } \csc A = \frac{41}{40}; \text{ find } \cos A.$$

$$14. \quad \text{Given } \sec A = 5; \text{ find } \cot A.$$

$$15. \quad \text{Given } \cos A = \frac{21}{29}; \text{ find } \csc A.$$

$$16. \quad \text{Given } \tan A = \frac{4\sqrt{2}}{7}; \text{ find } \sec A.$$

$$17. \quad \text{Given } \sin A = \frac{2}{7}; \text{ find } \text{vers } A.$$

 7. Functions of 45° .


Let ABC be an isosceles right triangle, C being the right angle, and sides AC and BC being each equal to 1.

$$\text{Then, } \angle A = 45^\circ, \text{ and } AB = \sqrt{AC^2 + BC^2} = \sqrt{1+1} = \sqrt{2}.$$

Whence, by definition,

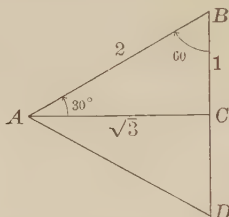
$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{1}{2} \sqrt{2}. \quad \sec 45^\circ = \sqrt{2}.$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{1}{2} \sqrt{2}. \quad \csc 45^\circ = \sqrt{2}.$$

$$\tan 45^\circ = 1. \quad \text{vers } 45^\circ = 1 - \frac{1}{2} \sqrt{2} = \frac{2 - \sqrt{2}}{2}.$$

$$\cot 45^\circ = 1. \quad \text{covers } 45^\circ = 1 - \frac{1}{2} \sqrt{2} = \frac{2 - \sqrt{2}}{2}.$$

8. Functions of 30° and 60° .



Let ABD be an equilateral triangle having each side equal to 2, and draw line AC perpendicular to BD .

By Geometry, $BC = \frac{1}{2} BD = 1$, $\angle BAC = \frac{1}{2} \angle BAD = 30^\circ$.

Also, $AC = \sqrt{AB^2 - BC^2} = \sqrt{4 - 1} = \sqrt{3}$.

Then from right triangle ABC , by definition,

$$\sin 30^\circ = \frac{1}{2} = \cos 60^\circ. \quad \cos 30^\circ = \frac{\sqrt{3}}{2} = \sin 60^\circ.$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{1}{3} \sqrt{3} = \cot 60^\circ. \quad \cot 30^\circ = \sqrt{3} = \tan 60^\circ.$$

$$\sec 30^\circ = \frac{2}{\sqrt{3}} = \frac{2}{3} \sqrt{3} = \csc 60^\circ. \quad \csc 30^\circ = 2 = \sec 60^\circ$$

$$\text{vers } 30^\circ = 1 - \frac{\sqrt{3}}{2} = \text{covers } 60^\circ.$$

$$\text{covers } 30^\circ = 1 - \frac{1}{2} = \frac{1}{2} = \text{vers } 60^\circ.$$

II. TRIGONOMETRIC FUNCTIONS OF ANGLES IN GENERAL.

9. In Geometry, we are, as a rule, concerned with angles less than two right angles; but in Trigonometry it is convenient to consider them as unrestricted in magnitude.

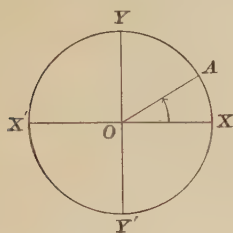


FIG. 1.

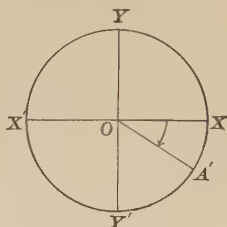


FIG. 2.

Let O be the centre, and XX' and YY' a pair of perpendicular diameters, of circle AY' ; OY being above, and OY' below, XX' , when OX is horizontal and extends to the right, and OX' to the left, of O .

Let radius OA (Fig. 1) start from the position OX , and revolve about point O as a pivot towards the position OY .

When OA coincides with OY , it has generated an angle of 90° ; when it coincides with OX' , of 180° ; with OY' , of 270° ; with OX , its first position, of 360° ; with OY again, of 450° ; and so on.

Hence, a meaning may be attached to a positive angle of any number of degrees.

10. We may also conceive of a *negative* angle of any number of degrees.

Thus, if a positive angle indicates revolution from the position OX towards OY , a negative angle may be taken as indicating revolution from the position OX in the *opposite* direction, towards OY' .

Thus, if radius OA' (Fig. 2) starts from the position OX , and revolves about point O as a pivot towards the position OY' , when it coincides with OY' , it has generated an angle of -90° ; when it coincides with OX' , of -180° ; with OY , of -270° ; and so on.

Note. It is immaterial which direction we consider the positive direction of rotation; but having at the outset adopted a certain direction as positive, our subsequent operations must be in accordance.

11. In generating a positive or negative angle of any number of degrees, the line from which the rotation is supposed to commence is called the *initial line* of the angle, and the final position of the rotating radius the *terminal line*.

12. To designate an angle, we always write first the letter at the extremity of the initial line.

Thus, in designating the angle formed by the lines OX and OA , if we regard OX as the initial line, we should call it XOA ; and if we regard OA as the initial line, we should call it AOX .

13. There are always two angles less than 360° in absolute value, one positive and the other negative, having the same initial and terminal line.

Thus there are formed by OX and OA' (Fig. 2) the positive angle XOA' between 270° and 360° , and the negative angle XOA' between 0° and -90° .

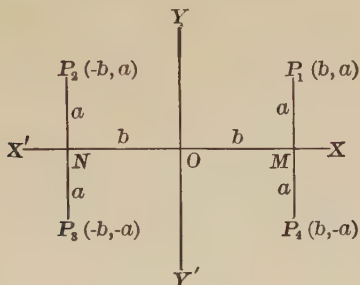
We shall distinguish between these angles by referring to them as "the positive angle XOA' ," and "the negative angle XOA' ," respectively.

14. Rectangular Co-ordinates.

Let XX' and YY' be two straight lines intersecting at right angles at O ; the letters being arranged as in the figures of § 9.

Let P_1 be any point in the plane of XX' and YY' , and draw line P_1M perpendicular to XX' .

Then OM and P_1M are called the *rectangular co-ordinates* of P_1 ; OM is called the *abscissa*, and P_1M the *ordinate*.



The lines of reference, XX' and YY' , are called the *axis of X* and the *axis of Y*, respectively, and O is called the *origin*.

It is customary to express the fact that the abscissa of a point is b , and its ordinate a , by saying that for the point in question $x = b$ and $y = a$; or, more concisely, we may refer to the point as "the point (b, a) ," where the first term in the parenthesis is understood to be the abscissa, and the second term the ordinate.

15. If, in the figure of § 14, M and N be points on OX and OX' , respectively, such that $OM = ON = b$, and lines P_1P_4 and P_2P_3 be drawn through M and N , respectively, perpendicular to XX' , making $P_1M = P_2N = P_3N = P_4M = a$, each of the points P_1 , P_2 , P_3 , and P_4 will have its abscissa equal to b , and its ordinate equal to a .

To avoid this ambiguity, abscissas measured to the *right* of O are considered *positive*, and to the *left*, *negative*; and ordinates measured *above* XX' are considered *positive*, and *below*, *negative*.

Then the co-ordinates of the points will be as follows:

$$P_1, (b, a); P_2, (-b, a); P_3, (-b, -a); P_4, (b, -a).$$

Note 1. It is understood, in the above convention with regard to signs, that *the figure is so placed that OX is horizontal, and extends to the right from O .*

Note 2. In all the figures of the present chapter, the small letters are understood as denoting *the lengths of the lines to which they are attached, without regard to their algebraic signs*; hence they always represent *positive quantities*.

16. If a point lies upon XX' , its ordinate is zero; and if it lies upon YY' , its abscissa is zero.

17. General Definitions of the Functions.

We will now give general definitions of the trigonometric functions, applicable to any angle whatever.

Construct axes in such a way that the initial line of the angle shall be the positive direction of the axis of X , and the vertex the origin.

From any point in the terminal line drop a perpendicular to the axis of X , and find the co-ordinates of this point.

Then, *the **sine** of the angle is the ratio of the **ordinate** of the point to its **distance** from the origin.*

*The **cosine** is the ratio of the **abscissa** to the **distance**.*

*The **tangent** is the ratio of the **ordinate** to the **abscissa**.*

*The **cotangent** is the ratio of the **abscissa** to the **ordinate**.*

*The **secant** is the ratio of the **distance** to the **abscissa**.*

*The **cosecant** is the ratio of the **distance** to the **ordinate**.*

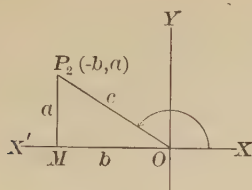
Note. The above definitions include those of § 2.

The definitions of the versed sine and covered sine, given in § 2, are sufficiently general to apply to any angle whatever.

18. We will now apply the definitions of § 17 in the following figures.

In each case, we construct axes in such a way that the initial line of the angle shall be the positive direction of the axis of X , and the vertex the origin.

I. Let XOP_2 be any angle between 90° and 180° .



Let P_2 be any point on the terminal line, and draw P_2M perpendicular to XX' ; let $P_2M = a$, $OM = b$, and $OP_2 = c$.

Then the co-ordinates of P_2 are $(-b, a)$.

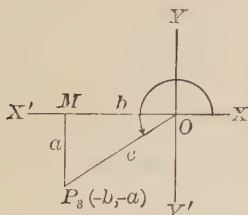
Whence, by definition,

$$\sin XOP_2 = \frac{a}{c} \qquad \cos XOP_2 = \frac{-b}{c} = -\frac{b}{c}$$

$$\tan XOP_2 = \frac{a}{-b} = -\frac{a}{b} \qquad \cot XOP_2 = \frac{-b}{a} = -\frac{b}{a}$$

$$\sec XOP_2 = \frac{c}{-b} = -\frac{c}{b} \qquad \csc XOP_2 = \frac{c}{a}$$

II. Let XOP_3 be any angle between 180° and 270° .



Let P_3 be any point on the terminal line, and draw P_3M perpendicular to XX' ; let $P_3M = a$, $OM = b$, and $OP_3 = c$.

Then the co-ordinates of P_3 are $(-b, -a)$.

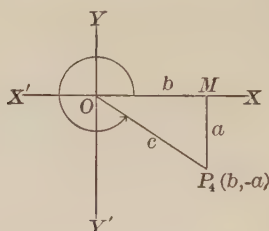
Whence, by definition,

$$\sin XOP_3 = \frac{-a}{c} = -\frac{a}{c}. \quad \cos XOP_3 = \frac{-b}{c} = -\frac{b}{c}.$$

$$\tan XOP_3 = \frac{-a}{-b} = \frac{a}{b}. \quad \cot XOP_3 = \frac{-b}{-a} = \frac{b}{a}.$$

$$\sec XOP_3 = \frac{c}{-b} = -\frac{c}{b}. \quad \csc XOP_3 = \frac{c}{-a} = -\frac{c}{a}.$$

III. Let XOP_4 be any angle between 270° and 360° .



Let P_4 be any point in the terminal line, and draw P_4M perpendicular to XX' ; let $P_4M = a$, $OM = b$, and $OP_4 = c$. Then the co-ordinates of P_4 are $(b, -a)$.

Whence, by definition,

$$\sin XOP_4 = \frac{-a}{c} = -\frac{a}{c}. \quad \cos XOP_4 = \frac{b}{c}.$$

$$\tan XOP_4 = \frac{-a}{b} = -\frac{a}{b}. \quad \cot XOP_4 = \frac{b}{-a} = -\frac{b}{a}.$$

$$\sec XOP_4 = \frac{c}{b}. \quad \csc XOP_4 = \frac{c}{-a} = -\frac{c}{a}.$$

19. It is evident that the terminal lines of any two angles which differ by a multiple of 360° are coincident, and hence the trigonometric functions of two such angles are identical.

Thus, the functions of 50° , 410° , 770° , -310° , etc., are identical.

20. If the initial line of an angle coincides with OX , and its terminal line lies between OX and OY , the angle is said to be in the *first quadrant*; if the terminal line lies between OY and OX' , the angle is said to be in the *second quadrant*; if between OX' and OY' , in the *third quadrant*; if between OY' and OX , in the *fourth quadrant*.

Thus, any positive angle between 0° and 90° , or 360° and 450° , or any negative angle between -270° and -360° , is in the first quadrant; any positive angle between 90° and 180° , or 450° and 540° , or any negative angle between -180° and -270° , is in the second quadrant.

21. It follows from the definitions of § 17 that, *for any angle in the first quadrant, all the functions are positive.*

It is also evident by inspection of the results of § 18 that:

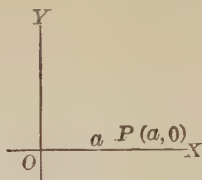
In the second quadrant, the sine and cosecant are positive, and the cosine, tangent, cotangent, and secant are negative.

In the third quadrant, the tangent and cotangent are positive, and the sine, cosine, secant, and cosecant are negative.

In the fourth quadrant, the cosine and secant are positive, and the sine, tangent, cotangent, and cosecant are negative.

It is customary to express the above in tabular form, as follows:

Functions.	First Quad.	Second Quad.	Third Quad.	Fourth Quad.
Sine and cosecant . . .	+	+	—	—
Cosine and secant . . .	+	—	—	+
Tangent and cotangent .	+	—	+	—

22. Functions of 0° and 360° .

The terminal line of 0° coincides with the initial line OX .

Let P be a point on OX such that $OP = a$.

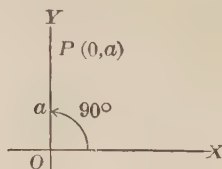
Then by § 16, the co-ordinates of P are $(a, 0)$.

Whence by definition,

$$\sin 0^\circ = \frac{0}{a} = 0. \quad \tan 0^\circ = \frac{0}{a} = 0. \quad \sec 0^\circ = \frac{a}{a} = 1.$$

$$\cos 0^\circ = \frac{a}{a} = 1. \quad \cot 0^\circ = \frac{a}{0} = \infty. \quad \csc 0^\circ = \frac{a}{0} = \infty.$$

By § 19, the functions of 360° are the same as those of 0° .

23. Functions of 90° .

For the angle 90° , OY is the terminal line.

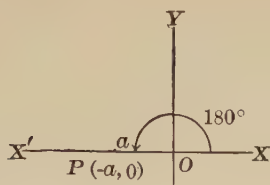
Let P be a point on OY , such that $OP = a$.

Then the co-ordinates of P are $(0, a)$.

Whence by definition,

$$\sin 90^\circ = \frac{a}{a} = 1. \quad \tan 90^\circ = \frac{a}{0} = \infty. \quad \sec 90^\circ = \frac{a}{0} = \infty.$$

$$\cos 90^\circ = \frac{0}{a} = 0. \quad \cot 90^\circ = \frac{0}{a} = 0. \quad \csc 90^\circ = \frac{a}{a} = 1.$$

24. Functions of 180° .


For the angle 180° , OX' is the terminal line.

Let P be a point on OX' , such that $OP = a$.

Then the co-ordinates of P are $(-a, 0)$.

Whence by definition,

$$\sin 180^\circ = \frac{0}{a} = 0.$$

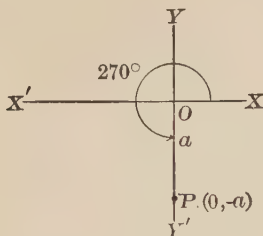
$$\cos 180^\circ = \frac{-a}{a} = -1$$

$$\tan 180^\circ = \frac{0}{-a} = 0.$$

$$\cot 180^\circ = \frac{-a}{0} = \infty.$$

$$\sec 180^\circ = \frac{a}{-a} = -1.$$

$$\csc 180^\circ = \frac{a}{0} = \infty.$$

25. Functions of 270°


For the angle 270° , OY' is the terminal line.

Let P be a point on OY' , such that $OP = a$.

Then the co-ordinates of P are $(0, -a)$.

Whence by definition,

$$\sin 270^\circ = \frac{-a}{a} = -1.$$

$$\cos 270^\circ = \frac{0}{a} = 0.$$

$$\tan 270^\circ = \frac{-a}{0} = \infty.$$

$$\cot 270^\circ = \frac{0}{-a} = 0.$$

$$\sec 270^\circ = \frac{a}{0} = \infty.$$

$$\csc 270^\circ = \frac{a}{-a} = -1.$$

Note. No absolute meaning can be attached to such a result as $\cot 0^\circ = \infty$; it merely signifies that as an angle approaches 0° , its cotangent increases without limit.

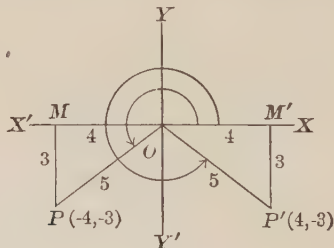
A similar interpretation must be given to the equations $\csc 0^\circ = \infty$, $\tan 90^\circ = \infty$, etc.

26. Given the value of one function of an angle, to find the values of the remaining functions. (Compare § 6.)

1. Given $\sin A = -\frac{3}{5}$; required the values of the remaining functions of A .

The example may be solved by a method similar to that of § 6; since the sine is the ratio of the ordinate to the distance, we may regard the point of reference as having its ordinate equal to -3 , and its distance equal to 5 .

There are *two* points, P and P' , which are 3 units below the axis of X , and distant 5 units from O .



There are then *two* angles, XOP and XOP' , in the third and fourth quadrants, respectively, either of which may be the angle A .

$$\text{Now, } OM = OM' = \sqrt{OP^2 - PM^2} = \sqrt{25 - 9} = 4.$$

Then co-ordinates of P are $(-4, -3)$, and of P' , $(4, -3)$.

Whence by definition :

Angle.	Cos.	Tan	Cot.	Sec.	Csc.
XOP	$-\frac{4}{5}$	$\frac{3}{4}$	$\frac{4}{3}$	$-\frac{5}{4}$	$-\frac{5}{3}$
XOP'	$\frac{4}{5}$	$-\frac{3}{4}$	$-\frac{4}{3}$	$\frac{5}{4}$	$-\frac{5}{3}$

Thus the two solutions to the problem are :

$$\cos A = \mp \frac{4}{5}, \tan A = \pm \frac{3}{4}, \cot A = \pm \frac{4}{3}, \sec A = \mp \frac{5}{4}, \csc A = -\frac{5}{3};$$

where the upper signs refer to XOP , and the lower signs to XOP' .

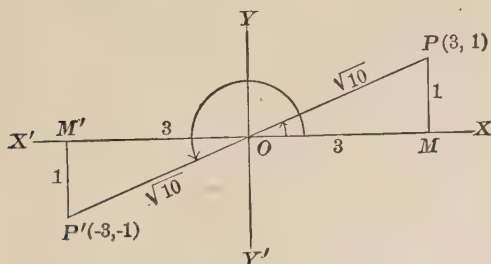
2. Given $\cot A = 3$; required the values of the remaining functions of A .

The equation may be written in the forms

$$\cot A = \frac{3}{1}, \text{ or } \cot A = \frac{-3}{-1}.$$

We may then regard the point of reference as having its abscissa equal to 3 and its ordinate equal to 1, or as having its abscissa equal to -3 and its ordinate equal to -1 .

There are *two angles*, XOP and XOP' , in the first and third quadrants, respectively, either of which satisfies the given condition.



$$\text{Then, } OP = OP' = \sqrt{OM^2 + PM^2} = \sqrt{9 + 1} = \sqrt{10}.$$

Whence by definition :

Angle.	Sin.	Cos.	Tan.	Sec.	Csc.
XOP	$\frac{1}{\sqrt{10}}$	$\frac{3}{\sqrt{10}}$	$\frac{1}{3}$	$\frac{\sqrt{10}}{3}$	$\sqrt{10}$
XOP'	$-\frac{1}{\sqrt{10}}$	$-\frac{3}{\sqrt{10}}$	$\frac{1}{3}$	$-\frac{\sqrt{10}}{3}$	$-\sqrt{10}$

Thus the two solutions are :

$$\sin A = \pm \frac{1}{\sqrt{10}}, \quad \cos A = \pm \frac{3}{\sqrt{10}}, \quad \tan A = \frac{1}{3},$$

$$\sec A = \pm \frac{\sqrt{10}}{3}, \quad \csc A = \pm \sqrt{10}.$$

Note. It must be clearly borne in mind, in examples like the above, that the "distance" is *always positive*.

EXAMPLES.

In each of the following, find the values of the remaining functions :

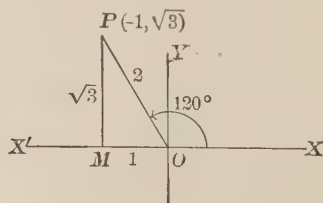
$$3. \sec A = \frac{5}{4}. \quad 7. \csc A = -\frac{25}{7}. \quad 11. \tan A = -7.$$

$$4. \cot A = -\frac{12}{5}. \quad 8. \tan A = \frac{9}{40}. \quad 12. \csc A = 3.$$

$$5. \sin A = \frac{15}{17}. \quad 9. \sec A = -\frac{7}{2}. \quad 13. \cos A = \frac{a}{b}.$$

$$6. \cos A = -\frac{21}{29}. \quad 10. \sin A = -\frac{1}{5}. \quad 14. \cot A = x.$$

27. Functions of 120° , 135° , 150° , etc.



Let OPM be a right triangle having OP , OM , and PM equal to 2, 1, and $\sqrt{3}$, respectively, and $\angle POM = 60^\circ$. (Compare § 8.)

Then $\angle XOP = 120^\circ$, and co-ordinates of P are $(-1, \sqrt{3})$.

Whence by definition,

$$\begin{aligned}\sin 120^\circ &= \frac{\sqrt{3}}{2}, & \cos 120^\circ &= -\frac{1}{2}. \\ \tan 120^\circ &= -\sqrt{3}. & \cot 120^\circ &= -\frac{1}{\sqrt{3}} = -\frac{1}{3}\sqrt{3}. \\ \sec 120^\circ &= -2. & \csc 120^\circ &= \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}.\end{aligned}$$

In like manner may be proved the remaining values given in the following table, which are left as exercises for the student:

Angle.	Sin.	Cos.	Tan.	Cot.	Sec.	Csc.
120°	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	-2	$\frac{2}{3}\sqrt{3}$
135°	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
210°	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2
225°	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	-2	$-\frac{2}{3}\sqrt{3}$
300°	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	2	$-\frac{2}{3}\sqrt{3}$
315°	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	-2

28. Functions of $(-A)$ in terms of those of A .

To prove the formulæ

$$\left. \begin{aligned}\sin(-A) &= -\sin A, & \cos(-A) &= \cos A, \\ \tan(-A) &= -\tan A, & \cot(-A) &= -\cot A, \\ \sec(-A) &= \sec A, & \csc(-A) &= -\csc A,\end{aligned}\right\} \quad (1)$$

for any value of A .

There may be four cases: A in the first quadrant (Fig. 1), A in the second quadrant (Fig. 2), A in the third quadrant (Fig. 3), or A in the fourth quadrant (Fig. 4).

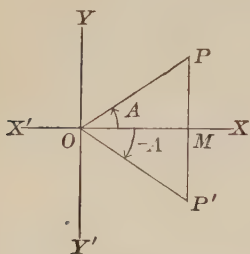


FIG. 1.

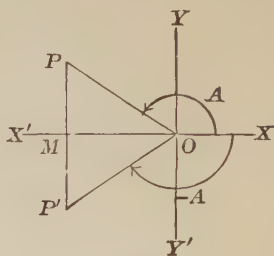


FIG. 2.

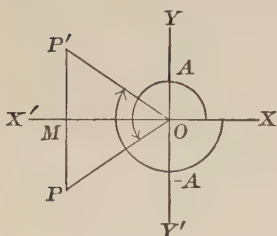


FIG. 3.

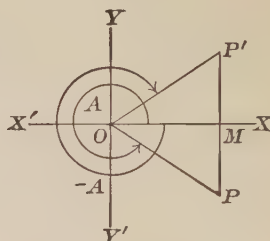


FIG. 4.

In each figure, let the positive angle XOP represent the angle A , and the negative angle XOP' the angle $-A$.

Draw PM perpendicular to XX' , and produce it to meet OP' at P' .

In right triangles OPM and $OP'M$, side OM is common, and $\angle POM = \angle P'OM$.

Then the triangles are equal, and $PM = P'M$ and $OP = OP'$.

Therefore, in each figure,

$$\text{abscissa } P' = \text{abscissa } P,$$

$$\text{ordinate } P' = - \text{ordinate } P,$$

and

$$\text{distance } P' = \text{distance } P.$$

$$\text{Then, } \sin(-A) = \frac{\text{ord. } P'}{\text{dist. } P'} = -\frac{\text{ord. } P}{\text{dist. } P} = -\sin A.$$

$$\cos(-A) = \frac{\text{abs. } P'}{\text{dist. } P'} = \frac{\text{abs. } P}{\text{dist. } P} = \cos A.$$

$$\tan(-A) = \frac{\text{ord. } P'}{\text{abs. } P'} = -\frac{\text{ord. } P}{\text{abs. } P} = -\tan A.$$

$$\cot(-A) = \frac{\text{abs. } P'}{\text{ord. } P'} = -\frac{\text{abs. } P}{\text{ord. } P} = -\cot A.$$

$$\sec(-A) = \frac{\text{dist. } P'}{\text{abs. } P'} = \frac{\text{dist. } P}{\text{abs. } P} = \sec A.$$

$$\csc(-A) = \frac{\text{dist. } P'}{\text{ord. } P'} = -\frac{\text{dist. } P}{\text{ord. } P} = -\csc A.$$

29. Functions of $(90^\circ + A)$ in terms of those of A .

To prove the formulæ

$$\left. \begin{aligned} \sin(90^\circ + A) &= \cos A, & \cos(90^\circ + A) &= -\sin A, \\ \tan(90^\circ + A) &= -\cot A, & \cot(90^\circ + A) &= -\tan A, \\ \sec(90^\circ + A) &= -\csc A, & \csc(90^\circ + A) &= \sec A, \end{aligned} \right\} \quad (2)$$

for any value of A .

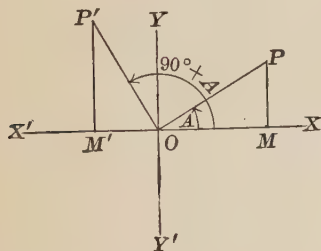


FIG. 1.

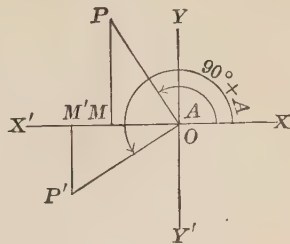


FIG. 2.

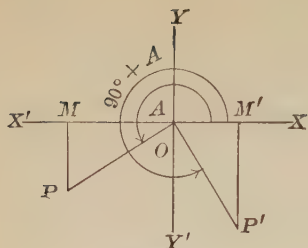


FIG. 3.

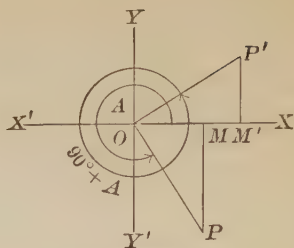


FIG. 4.

There may be four cases: A in the first quadrant (Fig. 1), A in the second quadrant (Fig. 2), A in the third quadrant (Fig. 3), or A in the fourth quadrant (Fig. 4).

In each figure, let the positive angle XOP represent the angle A , and the positive angle XOP' the angle $90^\circ + A$.

Take $OP' = OP$, and draw PM and $P'M'$ perpendicular to XX' .

Since OP is perpendicular to OP' , and OM to $P'M'$,

$$\angle POM = \angle OP'M'.$$

Then right triangles OPM and $OP'M'$ have the hypotenuse and an acute angle of one equal, respectively, to the hypotenuse and an acute angle of the other, and are equal.

Whence, $PM = OM'$ and $OM = P'M'$.

Therefore, in each figure,

$$\text{ordinate } P' = \text{abscissa } P,$$

$$\text{abscissa } P' = - \text{ordinate } P,$$

and

$$\text{distance } P' = \text{distance } P.$$

$$\text{Then, } \sin(90^\circ + A) = \frac{\text{ord. } P'}{\text{dist. } P'} = \frac{\text{abs. } P}{\text{dist. } P} = \cos A.$$

$$\cos(90^\circ + A) = \frac{\text{abs. } P'}{\text{dist. } P'} = - \frac{\text{ord. } P}{\text{dist. } P} = - \sin A.$$

$$\tan (90^{\circ} + A) = \frac{\text{ord. } P'}{\text{abs. } P'} = - \frac{\text{abs. } P}{\text{ord. } P} = - \cot A.$$

$$\cot (90^{\circ} + A) = \frac{\text{abs. } P'}{\text{ord. } P'} = - \frac{\text{ord. } P}{\text{abs. } P} = - \tan A.$$

$$\sec (90^{\circ} + A) = \frac{\text{dist. } P'}{\text{abs. } P'} = - \frac{\text{dist. } P}{\text{ord. } P} = - \csc A.$$

$$\csc (90^{\circ} + A) = \frac{\text{dist. } P'}{\text{ord. } P'} = \frac{\text{dist. } P}{\text{abs. } P} = \sec A.$$

30. The results of § 29 may be stated as follows :

The sine, cosine, tangent, cotangent, secant, and cosecant of any angle are equal, respectively, to the cosine, minus the sine, minus the cotangent, minus the tangent, minus the cosecant, and the secant, of an angle 90° less.

31. Functions of $(90^{\circ} - A)$ in terms of those of A .

By § 30, $\sin (90^{\circ} - A) = \cos (-A) = \cos A$ (§ 28).

$$\cos (90^{\circ} - A) = - \sin (-A) = \sin A.$$

$$\tan (90^{\circ} - A) = - \cot (-A) = \cot A.$$

$$\cot (90^{\circ} - A) = - \tan (-A) = \tan A.$$

$$\sec (90^{\circ} - A) = - \csc (-A) = \csc A.$$

$$\csc (90^{\circ} - A) = \sec (-A) = \sec A.$$

These formulæ were proved for acute angles in § 4.

32. Functions of $(180^{\circ} - A)$ in terms of those of A .

By § 30, $\sin (180^{\circ} - A) = \cos (90^{\circ} - A) = \sin A$ (§ 31)

$$\cos (180^{\circ} - A) = - \sin (90^{\circ} - A) = - \cos A.$$

$$\tan (180^{\circ} - A) = - \cot (90^{\circ} - A) = - \tan A.$$

$$\cot (180^{\circ} - A) = - \tan (90^{\circ} - A) = - \cot A.$$

$$\sec (180^{\circ} - A) = - \csc (90^{\circ} - A) = - \sec A.$$

$$\csc (180^{\circ} - A) = \sec (90^{\circ} - A) = \csc A.$$

33. By successive applications of the theorem of § 30, any function of a multiple of 90° , plus or minus A , may be expressed as a function of A .

1. Express $\sin (270^\circ + A)$ as a function of A .

By § 30, $\sin (270^\circ + A) = \cos (180^\circ + A) = -\sin (90^\circ + A) = -\cos A$.

If the multiple of 90° is greater than 270° , we may subtract 360° , or any multiple of 360° , from the angle, in accordance with § 19.

2. Express $\sec (990^\circ - A)$ as a function of A .

Subtracting twice 360° , or 720° , from the angle, we have

$$\sec (990^\circ - A) = \sec (270^\circ - A).$$

And by § 30, $\sec (270^\circ - A) = -\csc (180^\circ - A) = -\csc A$ (§ 32).

If the multiple of 90° is negative, we may add 360° , or any multiple of 360° , to the angle.

3. Express $\tan (-180^\circ + A)$ as a function of A .

Adding 360° to the angle, we have

$$\tan (-180^\circ + A) = \tan (180^\circ + A).$$

And by § 30, $\tan (180^\circ + A) = -\cot (90^\circ + A) = \tan A$.

EXAMPLES.

Express each of the following as a function of A :

- | | |
|--------------------------------------|--------------------------------------|
| 4. $\sin (180^\circ + A)$. | 11. $\csc (-90^\circ - A)$. |
| 5. $\cos (270^\circ - A)$. | 12. $\cot (-180^\circ + A)$. |
| 6. $\cot (450^\circ + A)$. | 13. $\sin (-630^\circ + A)$. |
| 7. $\csc (360^\circ - A)$. | 14. $\tan (-450^\circ - A)$. |
| 8. $\tan (540^\circ - A)$. | 15. $\cos (-900^\circ - A)$. |
| 9. $\sec (630^\circ + A)$. | 16. $\sin (810^\circ - A)$. |
| 10. $\tan (-270^\circ - A)$. | 17. $\csc (1080^\circ + A)$. |
| 18. $\sec (1260^\circ + A)$. | |

34. By means of the theorem of § 30, any function of any angle, positive or negative, may be expressed as a function of a certain acute angle.

1. Express $\sin 317^\circ$ as a function of an acute angle.

By § 30, $\sin 317^\circ = \cos 227^\circ = -\sin 137^\circ = -\cos 47^\circ$.

Since the complement of 47° is 43° , another form of the result is $-\sin 43^\circ$ (§ 4).

Note. As in the examples of § 33, 360° , or any multiple of 360° , may be added to, or subtracted from, the angle.

EXAMPLES.

Express each of the following as a function of an acute angle:

2. $\cos 322^\circ$. 4. $\sec 559^\circ$. 6. $\cot (-378^\circ)$.

3. $\tan 208^\circ$. 5. $\csc 803^\circ 45'$. 7. $\sin (-139^\circ 5')$.

It is evident from the above that any function of any angle can be expressed as a function of a certain acute angle *less than* 45° .

Express each of the following as a function of an acute angle less than 45° :

8. $\cot 155^\circ$. 10. $\sec 457^\circ$. 12. $\tan (-681^\circ)$.

9. $\sin 1138^\circ 36'$. 11. $\cos 496^\circ 20'$. 13. $\csc (-257^\circ)$.

14. Find the numerical value of $\csc (-210^\circ)$.

Adding 360° to the angle, we have

$$\csc (-210^\circ) = \csc 150^\circ.$$

And by § 30, $\csc 150^\circ = \sec 60^\circ = 2$ (§ 8).

Find the numerical values of the following:

15. $\cot 405^\circ$. 17. $\csc 600^\circ$. 19. $\cos (-420^\circ)$.

16. $\sin 480^\circ$. 18. $\tan 690^\circ$. 20. $\sec (-225^\circ)$.

III. GENERAL FORMULÆ.

35. It follows directly from the definitions of § 17 that if x is any angle,

$$\left. \begin{array}{lll} \sin x = \frac{1}{\csc x} & \tan x = \frac{1}{\cot x} & \sec x = \frac{1}{\cos x} \\ \cos x = \frac{1}{\sec x} & \cot x = \frac{1}{\tan x} & \csc x = \frac{1}{\sin x} \end{array} \right\} \quad (3)$$

36. To prove the formula

$$\tan x = \frac{\sin x}{\cos x} \quad (4)$$

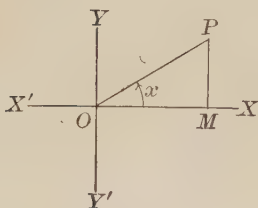


FIG. 1.

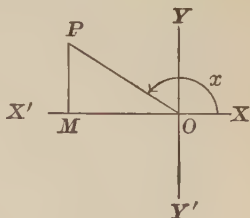


FIG. 2.

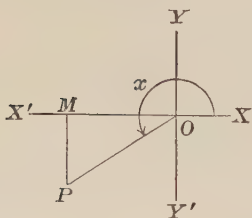


FIG. 3.

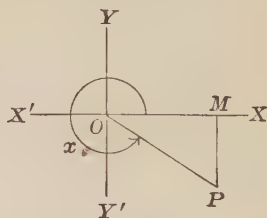


FIG. 4.

There may be four cases: x in the first quadrant (Fig. 1), x in the second quadrant (Fig. 2), x in the third quadrant (Fig. 3), or x in the fourth quadrant (Fig. 4).

In each case, let the positive angle XOP represent the angle x , and draw PM perpendicular to XX' .

Then in each figure, by the definitions of § 17,

$$\tan x = \frac{\text{ord. } P}{\text{abs. } P} = \frac{\frac{\text{ord. } P}{\text{dist. } P}}{\frac{\text{abs. } P}{\text{dist. } P}} = \frac{\sin x}{\cos x}.$$

37. *To prove the formula*

$$\cot x = \frac{\cos x}{\sin x}. \quad (5)$$

By § 35, $\cot x = \frac{1}{\tan x} = \frac{1}{\frac{\sin x}{\cos x}} (\S 36) = \frac{\cos x}{\sin x}.$

38. *To prove the formula*

$$\sin^2 x + \cos^2 x = 1. \quad (6)$$

Note. $\sin^2 x$ signifies $(\sin x)^2$; that is, the square of the sine of x .

There may be four cases: x in the first quadrant, x in the second quadrant, x in the third quadrant, or x in the fourth quadrant.

In each figure of § 36, we have by Geometry,

$$\overline{PM}^2 + \overline{OM}^2 = \overline{OP}^2.$$

Dividing by \overline{OP}^2 , $\frac{\overline{PM}^2}{\overline{OP}^2} + \frac{\overline{OM}^2}{\overline{OP}^2} = 1.$

But in each figure,

$$\frac{\overline{PM}^2}{\overline{OP}^2} = (\sin x)^2, \text{ and } \frac{\overline{OM}^2}{\overline{OP}^2} = (\cos x)^2;$$

whether $\sin x$ equals $+\frac{PM}{OP}$ or $-\frac{PM}{OP}$, its square is $\frac{\overline{PM}^2}{\overline{OP}^2}$

Whence, $\sin^2 x + \cos^2 x = 1.$

39. Formula (6) may be written in the forms

$$\sin^2 x = 1 - \cos^2 x, \text{ and } \cos^2 x = 1 - \sin^2 x.$$

40. To prove the formulæ

$$\sec^2 x = 1 + \tan^2 x, \quad (7)$$

and

$$\csc^2 x = 1 + \cot^2 x. \quad (8)$$

By (6),

$$1 = \cos^2 x + \sin^2 x. \quad (A)$$

$$\text{Dividing by } \cos^2 x, \quad \frac{1}{\cos^2 x} = 1 + \frac{\sin^2 x}{\cos^2 x}.$$

Whence by (3) and (4), $\sec^2 x = 1 + \tan^2 x$.

Again, dividing (A) by $\sin^2 x$, we have

$$\frac{1}{\sin^2 x} = 1 + \frac{\cos^2 x}{\sin^2 x}.$$

Whence by (3) and (5), $\csc^2 x = 1 + \cot^2 x$.

41. To express $\sin(x+y)$ and $\cos(x+y)$ in terms of the sines and cosines of x and y .

I. When x and y are acute.

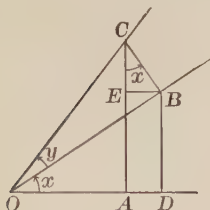


FIG. 1.

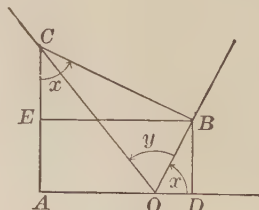


FIG. 2.

There may be two cases: $x+y$ acute (Fig. 1), and $x+y$ obtuse (Fig. 2).

In each figure, let $\angle DOB = x$ and $\angle BOC = y$.

Then,

$$\angle DOC = x + y.$$

From any point C in OC draw lines CA and CB perpendicular to OD and OB , respectively; also, draw lines BD and BE perpendicular to OD and AC , respectively.

Since EC is perpendicular to OD , and BC to OB , angles BCE and DOB are equal; that is, $\angle BCE = x$.

In either figure, by § 17,

$$\begin{aligned}\sin(x+y) &= \frac{AC}{OC} = \frac{BD+CE}{OC} = \frac{BD}{OC} + \frac{CE}{OC} \\ &= \frac{BD}{OB} \times \frac{OB}{OC} + \frac{CE}{BC} \times \frac{BC}{OC}.\end{aligned}$$

Then, $\sin(x+y) = \sin x \cos y + \cos x \sin y.$ (9)

Again, by § 17, in Fig. 1, $\cos(x+y) = \frac{OA}{OC},$

and in Fig. 2, $\cos(x+y) = -\frac{OA}{OC}.$

Then in either figure,

$$\begin{aligned}\cos(x+y) &= \frac{OD-BE}{OC} = \frac{OD}{OC} - \frac{BE}{OC} \\ &= \frac{OD}{OB} \times \frac{OB}{OC} - \frac{BE}{BC} \times \frac{BC}{OC}.\end{aligned}$$

Then, $\cos(x+y) = \cos x \cos y - \sin x \sin y.$ (10)

42. Formulæ (9) and (10) are very important, and it is necessary to prove them *for all values of x and y .*

They have already been proved when x and y are any two acute angles; or, what is the same thing, when they are any two angles in the *first quadrant*.

Now let a and b be any two angles in the first quadrant.

By § 29, $\sin[90^\circ + (a+b)] = \cos(a+b),$

and $\cos[90^\circ + (a+b)] = -\sin(a+b).$

Whence, by (9) and (10),

$$\sin[90^\circ + (a+b)] = \cos a \cos b - \sin a \sin b, \quad (\text{A})$$

and $\cos[90^\circ + (a+b)] = -\sin a \cos b - \cos a \sin b. \quad (\text{B})$

By § 29, $\cos a = \sin (90^\circ + a)$, and $-\sin a = \cos (90^\circ + a)$.

Then, (A) and (B) may be written in the forms

$$\sin [(90^\circ + a) + b] = \sin (90^\circ + a) \cos b + \cos (90^\circ + a) \sin b,$$

$$\cos [(90^\circ + a) + b] = \cos (90^\circ + a) \cos b - \sin (90^\circ + a) \sin b;$$

which are in accordance with (9) and (10).

But $90^\circ + a$ is an angle in the second quadrant.

Therefore, (9) and (10) hold when one of the angles is in the second quadrant, and the other in the first.

In like manner, by supposing a to be any angle in the first quadrant, and b any angle in the second, (9) and (10) may be proved to hold when both angles are in the second quadrant.

Again, by supposing a and b to be any two angles in the second quadrant, (9) and (10) may be proved to hold when one angle is in the second quadrant and the other in the third; and so on.

Hence, (9) and (10) hold for any values of x and y whatever, positive or negative.

43. Putting, in (9) and (10), $-y$ in place of y ,

$$\begin{aligned}\sin (x - y) &= \sin x \cos (-y) + \cos x \sin (-y) \\ &= \sin x \cos y + \cos x (-\sin y), \text{ by § 28,} \\ &= \sin x \cos y - \cos x \sin y.\end{aligned}\tag{11}$$

$$\begin{aligned}\cos (x - y) &= \cos x \cos (-y) - \sin x \sin (-y) \\ &= \cos x \cos y - \sin x (-\sin y) \\ &= \cos x \cos y + \sin x \sin y.\end{aligned}\tag{12}$$

44. By (4),

$$\begin{aligned}\tan (x + y) &= \frac{\sin (x + y)}{\cos (x + y)} \\ &= \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}, \text{ by (9) and (10).}\end{aligned}$$

Dividing each term of the fraction by $\cos x \cos y$,

$$\begin{aligned}\tan(x+y) &= \frac{\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y}} \\ &= \frac{\tan x + \tan y}{1 - \tan x \tan y}.\end{aligned}\tag{13}$$

In like manner, we may prove

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.\tag{14}$$

$$\begin{aligned}\text{Again, by (5), } \cot(x+y) &= \frac{\cos(x+y)}{\sin(x+y)} \\ &= \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y}.\end{aligned}$$

Dividing each term of the fraction by $\sin x \sin y$,

$$\begin{aligned}\cot(x+y) &= \frac{\frac{\cos x \cos y}{\sin x \sin y} - \frac{\sin x \sin y}{\sin x \sin y}}{\frac{\sin x \cos y}{\sin x \sin y} + \frac{\cos x \sin y}{\sin x \sin y}} \\ &= \frac{\cot x \cot y - 1}{\cot y + \cot x}.\end{aligned}\tag{15}$$

In like manner, we may prove

$$\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}.\tag{16}$$

45 From (9), (10), (11), and (12), we have

$$\sin(a+b) = \sin a \cos b + \cos a \sin b.\tag{A}$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b.\tag{B}$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.\tag{C}$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b.\tag{D}$$

Adding and subtracting (A) and (B), and then (C) and (D),

$$\sin(a+b) + \sin(a-b) = 2 \sin a \cos b.$$

$$\sin(a+b) - \sin(a-b) = 2 \cos a \sin b.$$

$$\cos(a+b) + \cos(a-b) = 2 \cos a \cos b.$$

$$\cos(a+b) - \cos(a-b) = -2 \sin a \sin b.$$

Let $a+b=x$, and $a-b=y$.

Adding, $2a = x+y$, or $a = \frac{1}{2}(x+y)$.

Subtracting, $2b = x-y$, or $b = \frac{1}{2}(x-y)$.

Substituting these values, we have

$$\sin x + \sin y = 2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y). \quad (17)$$

$$\sin x - \sin y = 2 \cos \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y). \quad (18)$$

$$\cos x + \cos y = 2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y). \quad (19)$$

$$\cos x - \cos y = -2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y). \quad (20)$$

46. By (17) and (18), we have

$$\begin{aligned} \frac{\sin x + \sin y}{\sin x - \sin y} &= \frac{2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)}{2 \cos \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y)} \\ &= \tan \frac{1}{2}(x+y) \cot \frac{1}{2}(x-y) \\ &= \frac{\tan \frac{1}{2}(x+y)}{\tan \frac{1}{2}(x-y)}, \text{ by } \S 35. \end{aligned} \quad (21)$$

47. Functions of $2x$.

Putting $y=x$ in (9), we have

$$\begin{aligned} \sin 2x &= \sin x \cos x + \cos x \sin x \\ &= 2 \sin x \cos x. \end{aligned} \quad (22)$$

Putting $y=x$ in (10), we obtain

$$\begin{aligned} \cos 2x &= \cos x \cos x - \sin x \sin x \\ &= \cos^2 x - \sin^2 x. \end{aligned} \quad (23)$$

We also have by § 39,

$$\cos 2x = (1 - \sin^2 x) - \sin^2 x = 1 - 2 \sin^2 x. \quad (24)$$

$$\cos 2x = \cos^2 x - (1 - \cos^2 x) = 2 \cos^2 x - 1. \quad (25)$$

Putting $y = x$ in (13) and (15),

$$\tan 2x = \frac{\tan x + \tan x}{1 - \tan x \tan x} = \frac{2 \tan x}{1 - \tan^2 x}. \quad (26)$$

$$\cot 2x = \frac{\cot x \cot x - 1}{\cot x + \cot x} = \frac{\cot^2 x - 1}{2 \cot x}. \quad (27)$$

48. Functions of $\frac{1}{2}x$.

From (24) and (25) we have, by transposition,

$$2 \sin^2 x = 1 - \cos 2x, \text{ and } 2 \cos^2 x = 1 + \cos 2x.$$

Putting $\frac{1}{2}x$ in place of x , and therefore x in place of $2x$, we have

$$2 \sin^2 \frac{1}{2}x = 1 - \cos x, \quad (28)$$

$$2 \cos^2 \frac{1}{2}x = 1 + \cos x. \quad (29)$$

Again, putting $\frac{1}{2}x$ in place of x in (22),

$$2 \sin \frac{1}{2}x \cos \frac{1}{2}x = \sin x. \quad (A)$$

Dividing (28) by (A),

$$\frac{2 \sin^2 \frac{1}{2}x}{2 \sin \frac{1}{2}x \cos \frac{1}{2}x} = \frac{1 - \cos x}{\sin x}.$$

$$\text{Whence, by (4),} \quad \tan \frac{1}{2}x = \frac{1 - \cos x}{\sin x}. \quad (30)$$

Dividing (29) by (A),

$$\frac{2 \cos^2 \frac{1}{2}x}{2 \sin \frac{1}{2}x \cos \frac{1}{2}x} = \frac{1 + \cos x}{\sin x}.$$

$$\text{Whence, by (5),} \quad \cot \frac{1}{2}x = \frac{1 + \cos x}{\sin x}. \quad (31)$$

EXERCISES.

49. 1. Prove the relation $\sec^2 x \csc^2 x = \sec^2 x + \csc^2 x$.

$$\begin{aligned} \text{By (3), } \sec^2 x \csc^2 x &= \frac{1}{\cos^2 x \sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x \sin^2 x}, \text{ by (6)} \\ &= \frac{\sin^2 x}{\cos^2 x \sin^2 x} + \frac{\cos^2 x}{\cos^2 x \sin^2 x} \\ &= \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} = \sec^2 x + \csc^2 x. \end{aligned}$$

2. Prove the relation $\frac{\sin 5x - \sin x}{\cos 5x + \cos x} = \tan 2x$.

By (18) and (19),

$$\frac{\sin 5x - \sin x}{\cos 5x + \cos x} = \frac{2 \cos \frac{1}{2}(5x + x) \sin \frac{1}{2}(5x - x)}{2 \cos \frac{1}{2}(5x + x) \cos \frac{1}{2}(5x - x)} = \frac{\sin 2x}{\cos 2x} = \tan 2x.$$

3. Prove the relation $\frac{\tan(x+y) - \tan x}{1 + \tan(x+y) \tan x} = \tan y$.

$$\text{By (14), } \frac{\tan(x+y) - \tan x}{1 + \tan(x+y) \tan x} = \tan[(x+y) - x] = \tan y.$$

4. Prove the relation $\sin 3x = 3 \sin x - 4 \sin^3 x$.

$$\text{By (18), } \sin 3x - \sin x = 2 \cos \frac{1}{2}(3x + x) \sin \frac{1}{2}(3x - x).$$

$$\begin{aligned} \text{Then, } \sin 3x &= \sin x + 2 \cos 2x \sin x \\ &= \sin x + 2(1 - 2 \sin^2 x) \sin x, \text{ by (24)} \\ &= \sin x + 2 \sin x - 4 \sin^3 x = 3 \sin x - 4 \sin^3 x. \end{aligned}$$

The artifice used above is advantageous in finding the sine or cosine of any *odd* multiple of x .

Prove the following relations:

$$5. \frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y}.$$

$$6. \frac{\cos(x+y)}{\cos(x-y)} = \frac{\cot x \cot y - 1}{\cot x \cot y + 1}.$$

$$7. \frac{\cos x + \cos y}{\cos x - \cos y} = -\cot \frac{1}{2}(x+y) \cot \frac{1}{2}(x-y).$$

$$8. \frac{\sin 3x + \sin x}{\cos 3x + \cos x} = \tan 2x.$$

$$9. \sec^2 A + \tan^2 B = \sec^2 B + \tan^2 A.$$

$$10. \tan x + \tan y = \frac{\sin(x+y)}{\cos x \cos y}.$$

$$11. \frac{\tan(A+B) + \tan(A-B)}{1 - \tan(A+B)\tan(A-B)} = \tan 2A.$$

$$12. \cos x \cos(x-y) + \sin x \sin(x-y) = \cos y.$$

$$13. \sin 5x \cos 2x - \cos 5x \sin 2x = \sin 3x.$$

$$14. \frac{\sin 3x - \sin 5x}{\cos 3x - \cos 5x} = -\cot 4x.$$

$$15. \frac{\cot x + \tan x}{\cot x - \tan x} = \sec 2x.$$

$$16. \cos 3x = 4 \cos^3 x - 3 \cos x.$$

$$17. \tan^2 \frac{1}{2}x = \frac{1 - \cos x}{1 + \cos x}.$$

$$18. \sin(x+y+z) = \sin x \cos y \cos z + \cos x \sin y \cos z \\ + \cos x \cos y \sin z - \sin x \sin y \sin z.$$

$$19. \cos(x+y+z) = \cos x \cos y \cos z - \sin x \sin y \cos z \\ - \sin x \cos y \sin z - \cos x \sin y \sin z.$$

$$20. \tan(x+y+z) = \frac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - \tan x \tan y - \tan y \tan z - \tan z \tan x}.$$

21. Prove the relation of Ex. 4 by putting $x = 2x$, and $y = x$, in (9).

Prove the relations:

$$22. \tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}.$$

$$23. \sin(x+y) \sin(x-y) = \sin^2 x - \sin^2 y.$$

$$24. \cos(x+y) \cos(x-y) = \cos^2 x - \sin^2 y.$$

$$25. \sin 2x = \frac{2 \tan x}{1 + \tan^2 x}.$$

$$26. \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}.$$

$$27. 4 \cos x \cos (60^\circ + x) \cos (60^\circ - x) = \cos 3x.$$

$$28. \tan (45^\circ + x) - \tan (45^\circ - x) = 2 \tan 2x.$$

$$29. \cos^2(x + y) - \sin^2 x = \cos(2x + y) \cos y.$$

$$30. \cot A - \cot 2A = \csc 2A.$$

$$31. \frac{\sin 4x + \sin 3x}{\cos 3x - \cos 4x} = \cot \frac{1}{2}x.$$

$$32. \frac{\sin 50^\circ - \sin 10^\circ}{\cos 50^\circ - \cos 10^\circ} = -\sqrt{3}.$$

$$33. \cos 80^\circ + \cos 40^\circ = \cos 20^\circ.$$

$$34. \sin 4x = 4 \sin x \cos x - 8 \sin^3 x \cos x.$$

$$35. \cos 4x = 1 - 8 \cos^2 x + 8 \cos^4 x.$$

$$36. \cos 5x = 5 \cos x - 20 \cos^3 x + 16 \cos^5 x.$$

$$37. \cos(2x + y) + 2 \sin x \sin(x + y) = \cos y.$$

38. By putting $x = 45^\circ$ and $y = 30^\circ$ in (11) and (12), prove

$$\sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}, \quad \cos 15^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

39. By putting $x = 30^\circ$ in (30) and (31), prove

$$\tan 15^\circ = 2 - \sqrt{3}, \quad \cot 15^\circ = 2 + \sqrt{3}.$$

40. By putting $x = 15^\circ$ in (3), and using the results of Ex. 38, prove

$$\sec 15^\circ = \sqrt{6} - \sqrt{2}, \quad \csc 15^\circ = \sqrt{6} + \sqrt{2}.$$

41. By putting $x = 45^\circ$ in (28) and (29), prove

$$\sin 22\frac{1}{2}^\circ = \frac{1}{2}\sqrt{2 - \sqrt{2}}, \quad \cos 22\frac{1}{2}^\circ = \frac{1}{2}\sqrt{2 + \sqrt{2}}.$$

42. By putting $x = 45^\circ$ in (30) and (31), prove

$$\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1, \quad \cot 22\frac{1}{2}^\circ = \sqrt{2} + 1.$$

43. By putting $x = 22\frac{1}{2}^\circ$ in (7) and (8), and using the results of Ex. 42, prove

$$\sec 22\frac{1}{2}^\circ = \sqrt{4 - 2\sqrt{2}}, \quad \csc 22\frac{1}{2}^\circ = \sqrt{4 + 2\sqrt{2}}.$$

Prove the relations :

44. $\cos^4 x - \sin^4 x = \cos 2x.$

45. $\frac{\csc x + \cot x}{\csc x - \cot x} = \cot^2 \frac{1}{2}x.$

46. $\sin(90^\circ + A) + \sin(210^\circ + A) + \sin(210^\circ - A) = 0.$

47. $(\sin x + \sin y)^2 + (\cos x + \cos y)^2 = 4 \cos^2 \frac{x-y}{2}.$

48. $\tan 2x \cot x - 1 = \sec 2x.$

49. $\tan x \tan(60^\circ + x) \tan(120^\circ + x) = -\tan 3x.$

50. $\frac{1 + \sin 2x}{\cos 2x} = \frac{\cos x + \sin x}{\cos x - \sin x}.$

51. $\tan 4x = \frac{4 \tan x - 4 \tan^3 x}{1 - 6 \tan^2 x + \tan^4 x}.$

52. $\frac{1 - \sin x}{\cos x} = \tan(45^\circ - \frac{1}{2}x).$

53. $\frac{2 \tan(45^\circ - x)}{1 + \tan^2(45^\circ - x)} = \cos 2x.$

54. Prove the first result of Ex. 43 by putting $x = 22\frac{1}{2}^\circ$ in (3), and using the result of Ex. 41.

IV. MISCELLANEOUS THEOREMS.

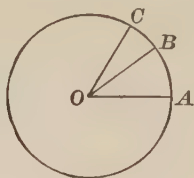
50. Circular Measure of an Angle.

An angle is measured by finding its ratio to another angle, adopted arbitrarily as the unit of measure.

The usual unit of measure for angles is the degree, which is an angle equal to the ninetieth part of a right angle.

Another method of measuring angles, and one of great importance, is known as the *Circular Method*; in which the unit of measure is *the angle at the centre of a circle subtended by an arc whose length is equal to the radius*.

Note. The unit of circular measure is called a *radian*.



Thus, let $\angle AOB$ be any central angle, and $\angle AOC$ the unit of circular measure; that is, the angle at the centre subtended by an arc whose length is equal to OA .

Then, circular measure $\angle AOB = \frac{\angle AOB}{\angle AOC}$.

But by Geometry, $\frac{\angle AOB}{\angle AOC} = \frac{\text{arc } AB}{\text{arc } AC} = \frac{\text{arc } AB}{OA}$.

Whence, circular measure $\angle AOB = \frac{\text{arc } AB}{OA}$.

That is, *the circular measure of an angle is the ratio of its subtending arc to the radius of the circle*.

51. By § 50, the circular measure of a right angle is the ratio of one-fourth the circumference to the radius.

But if R denotes the radius, the circumference is $2\pi R$.

Whence, circular measure $90^\circ = \frac{\frac{1}{4} \text{ of } 2\pi R}{R} = \frac{\pi}{2}$.

It follows from the above that the circular measure of 180° is π ; of 60° , $\frac{\pi}{3}$; of 45° , $\frac{\pi}{4}$; etc.

That is, *an angle expressed in degrees may be reduced to circular measure by finding its ratio to 180° , and multiplying the result by π .*

Thus, since 115° is $\frac{23}{36}$ of 180° , the circular measure of 115° is $\frac{23\pi}{36}$.

52. Conversely, *an angle expressed in circular measure may be reduced to degrees by multiplying by 180° and dividing by π ; or, more briefly, by substituting 180° for π .*

Thus, $\frac{7\pi}{15} = \frac{7}{15}$ of $180^\circ = 84^\circ$.

53. In the circular method, such expressions may occur as "the angle $\frac{2}{3}$," "the angle 1," etc.

These refer to the unit of circular measure; thus, the angle $\frac{2}{3}$ signifies an angle whose subtending arc is two-thirds of the radius.

The angle 1 signifies the angle whose subtending arc is equal to the radius, or the unit of circular measure.

The angle 1, reduced to degrees by the first rule of § 52, gives

$$\frac{180^\circ}{\pi} = \frac{180^\circ}{3.14159 \dots} = 57.2958^\circ, \text{ approximately.}$$

Then the rule of § 52 may be modified as follows:

An angle expressed in circular measure may be reduced to degrees by multiplying by 57.2958° .

Thus, the angle $\frac{2}{3}$

$$= \frac{2}{3} \times 57.2958^\circ = 38.1972^\circ = 38^\circ 11' 49.92''.$$

EXAMPLES.

54. Express each of the following in circular measure :

1. 120° . 3. $67^\circ 30'$. 5. $86^\circ 24'$. 7. $163^\circ 7' 30''$.
 2. 315° . 4. $146^\circ 15'$. 6. $53^\circ 20'$. 8. $88^\circ 53' 20''$

Express each of the following in degree measure :

9. $\frac{5\pi}{6}$. 11. $\frac{14\pi}{81}$. 13. $\frac{3}{2}$. 15. $\frac{5}{3}$.
 10. $\frac{11\pi}{24}$. 12. $\frac{23\pi}{64}$. 14. $\frac{1}{4}$. 16. $\frac{2}{5}$.

55. Inverse Trigonometric Functions.

The expression $\sin^{-1}x$, called the *inverse sine* of x , or the *anti-sine* of x , signifies *the angle whose sine is x* .

Thus, the statement that the sine of the angle x is equal to y may be expressed in either of the ways

$$\sin x = y, \text{ or } x = \sin^{-1}y.$$

In like manner, $\cos^{-1}x$ signifies the angle whose cosine is x ; $\tan^{-1}x$, the angle whose tangent is x ; etc.

Note. The student must be careful not to confuse the above notation with the *exponent* -1 ; the -1 power of $\sin x$ is expressed $(\sin x)^{-1}$, and not $\sin^{-1}x$.

It is evident that the sine of the angle whose sine is x is x ; that is, $\sin(\sin^{-1}x) = x$.

In like manner, $\cos(\cos^{-1}x) = x$; $\tan(\tan^{-1}x) = x$; etc.

56. By aid of the principles of § 55, we may derive from any formula involving direct functions a relation between inverse functions.

1. From the formula $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$, prove

$$\tan^{-1}a + \tan^{-1}b = \tan^{-1} \frac{a+b}{1-ab}.$$

Let $\tan x = a$, and $\tan y = b$.

Then by § 55, $x = \tan^{-1} a$, and $y = \tan^{-1} b$.

Substituting these values in the given formula,

$$\tan(\tan^{-1} a + \tan^{-1} b) = \frac{a + b}{1 - ab}.$$

Whence, $\tan^{-1} a + \tan^{-1} b = \tan^{-1} \frac{a + b}{1 - ab}$.

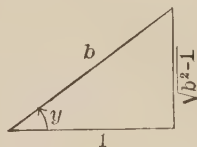
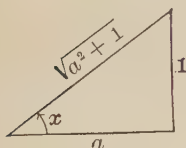
2. Prove the relation $\cot^{-1} a - \sec^{-1} b = \cos^{-1} \frac{a + \sqrt{b^2 - 1}}{b \sqrt{a^2 + 1}}$.

Let $\cot^{-1} a = x$, and $\sec^{-1} b = y$.

Then, $\cot x = a$, and $\sec y = b$.

Now, $\cos(x - y) = \cos x \cos y + \sin x \sin y$. (A)

To find the sines and cosines of x and y , we use the method of § 6.



In the right triangle containing angle x , the adjacent side is a , and the opposite side 1 ; then, the hypotenuse is $\sqrt{a^2 + 1}$.

In the right triangle containing angle y , the hypotenuse is b , and the adjacent side 1 ; then, the opposite side is $\sqrt{b^2 - 1}$.

Substituting the values of $\cos x$, $\cos y$, $\sin x$, and $\sin y$ in (A), we have

$$\cos(x - y) = \frac{a}{\sqrt{a^2 + 1}} \cdot \frac{1}{b} + \frac{1}{\sqrt{a^2 + 1}} \cdot \frac{\sqrt{b^2 - 1}}{b} = \frac{a + \sqrt{b^2 - 1}}{b \sqrt{a^2 + 1}}.$$

Whence, $x - y$ or $\cot^{-1} a - \sec^{-1} b = \cos^{-1} \frac{a + \sqrt{b^2 - 1}}{b \sqrt{a^2 + 1}}$.

EXAMPLES.

3. From $\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}$, prove $2 \cot^{-1} a = \cot^{-1} \frac{a^2 - 1}{2a}$.

4. From $\cos 2x = 1 - 2 \sin^2 x$, prove

$$2 \sin^{-1} a = \cos^{-1}(1 - 2a^2).$$

5. From $\sin 2x = 2 \sin x \cos x$, prove

$$2 \cos^{-1} a = \sin^{-1}(2a\sqrt{1-a^2}).$$

6. From $\cos(x+y) = \cos x \cos y - \sin x \sin y$, prove

$$\cos^{-1} a + \cos^{-1} b = \cos^{-1}(ab - \sqrt{1-a^2}\sqrt{1-b^2}).$$

Prove the following relations:

$$7. \cot^{-1} a + \cot^{-1} b = \cot^{-1} \frac{ab-1}{a+b}.$$

$$8. 2 \cos^{-1} a = \cos^{-1}(2a^2 - 1).$$

$$9. \sin^{-1} a - \sin^{-1} b = \sin^{-1}(a\sqrt{1-b^2} - b\sqrt{1-a^2}).$$

$$10. 3 \sin^{-1} a = \sin^{-1}(3a - 4a^3). \quad (\text{Ex. 4, p. 34.})$$

$$11. \tan^{-1} a + \cot^{-1} b = \tan^{-1} \frac{ab+1}{b-a}.$$

$$12. \cot^{-1}(a-b) - \cot^{-1}(a+b) = \cot^{-1} \frac{a^2-b^2+1}{2b}.$$

$$13. \tan^{-1} a = \sin^{-1} \frac{a}{\sqrt{a^2+1}}.$$

$$14. \csc^{-1} a = \cos^{-1} \frac{\sqrt{a^2-1}}{a}.$$

$$15. \sin^{-1} a + \cos^{-1} b = \tan^{-1} \frac{ab + \sqrt{1-a^2}\sqrt{1-b^2}}{b\sqrt{1-a^2} - a\sqrt{1-b^2}}.$$

$$16. \sec^{-1} a - \csc^{-1} b = \cos^{-1} \frac{\sqrt{a^2-1} + \sqrt{b^2-1}}{ab}.$$

$$17. \tan^{-1} a + \cos^{-1} \frac{1}{a} = \sin^{-1} \frac{a + \sqrt{a^2-1}}{a\sqrt{a^2+1}}.$$

$$18. 2 \sin^{-1} a = \tan^{-1} \frac{2a\sqrt{1-a^2}}{1-2a^2}.$$

$$19. \tan^{-1} \frac{a}{a-1} - \tan^{-1} \frac{a+1}{a} = \tan^{-1} \frac{1}{2a^2}.$$

$$20. \quad 2 \sec^{-1} a = \cot^{-1} \frac{2 - a^2}{2\sqrt{a^2 - 1}}.$$

57. The following table expresses the value of each of the six principal functions in terms of the other five:



The reciprocal forms were proved in § 35.

The others may be derived by aid of §§ 35, 36, 37, 39, and 40, and are left as exercises for the pupil.

As an illustration, we will prove the formula

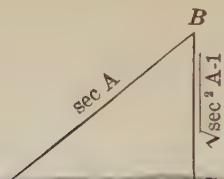
$$\cos A = \frac{\sqrt{\csc^2 A - 1}}{\csc A}.$$

By § 39,

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{1}{\csc^2 A}} = \frac{\sqrt{\csc^2 A - 1}}{\csc A}.$$

They may also be conveniently proved by the method of § 6; thus, let it be required to prove the formula for each of the other functions in terms of the secant.

$$\text{We have} \quad \sec A = \frac{\sec A}{1}.$$



58. Line Values of the Functions.

Let XOB be any angle.

With O as a centre, and a radius equal to 1, describe circumference AB , cutting OX at A , OB at B , and OY at C .

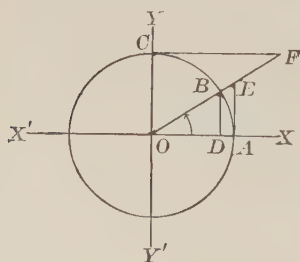


FIG. 1.

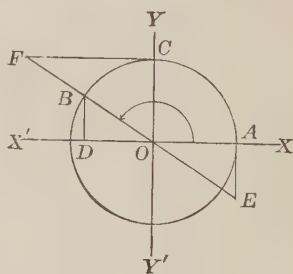


FIG. 2.

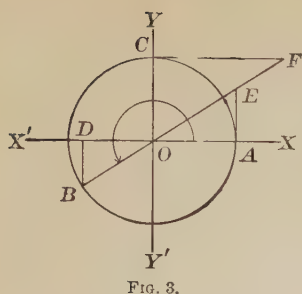


FIG. 3.

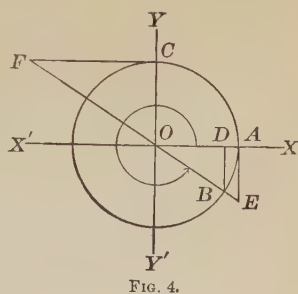


FIG. 4.

Draw line BD perpendicular to XX' ; also, lines AE and CF perpendicular to OX and OY , respectively, meeting OB , or OB produced, at E and F , respectively.

Then by § 17, the functions of $\angle XO B$ are :

	<i>Sin.</i>	<i>Cos.</i>	<i>Tan.</i>	<i>Cot.</i>	<i>Sec.</i>	<i>Csc.</i>
Fig. 1.	$\frac{BD}{OB}$	$\frac{OD}{OB}$	$\frac{BD}{OD}$	$\frac{OD}{BD}$	$\frac{OB}{OD}$	$\frac{OB}{BD}$
Fig. 2.	$\frac{BD}{OB}$	$-\frac{OD}{OB}$	$-\frac{BD}{OD}$	$-\frac{OD}{BD}$	$-\frac{OB}{OD}$	$\frac{OB}{BD}$
Fig. 3.	$-\frac{BD}{OB}$	$-\frac{OD}{OB}$	$\frac{BD}{OD}$	$\frac{OD}{BD}$	$-\frac{OB}{OD}$	$-\frac{OB}{BD}$
Fig. 4.	$-\frac{BD}{OB}$	$\frac{OD}{OB}$	$-\frac{BD}{OD}$	$-\frac{OD}{BD}$	$\frac{OB}{OD}$	$-\frac{OB}{BD}$

Now right triangles OBD , OEA , and OCF are similar, since their sides are parallel each to each.

Then, since $OA = OC = 1$, we have

$$\begin{aligned} \frac{BD}{OD} &= \frac{AE}{OA} = AE, & \frac{OB}{OD} &= \frac{OE}{OA} = OE, \\ \frac{OD}{BD} &= \frac{CF}{OC} = CF, & \frac{OB}{BD} &= \frac{OF}{OC} = OF. \end{aligned}$$

Whence, since $OB = 1$, the functions of $\angle XO B$ are :

	<i>Sin.</i>	<i>Cos.</i>	<i>Tan.</i>	<i>Cot.</i>	<i>Sec.</i>	<i>Csc.</i>
Fig. 1.	BD	OD	AE	CF	OE	OF
Fig. 2.	BD	$-OD$	$-AE$	$-CF$	$-OE$	OF
Fig. 3.	$-BD$	$-OD$	AE	CF	$-OE$	$-OF$
Fig. 4.	$-BD$	OD	$-AE$	$-CF$	OE	$-OF$

That is, *if the radius of the circle is 1,*

The *sine* is the perpendicular drawn to XX' from the intersection of the circumference with the terminal line.

The *cosine* is the line drawn from the centre to the foot of the sine.

The *tangent* is that portion of the geometrical tangent to the circle at the intersection of its circumference with OX included between OX and the terminal line, produced if necessary.

The *cotangent* is that portion of the geometrical tangent to the circle at the intersection of its circumference with OY included between OY and the terminal line, produced if necessary.

The *secant* is that portion of the terminal line, or terminal line produced, included between the centre and the tangent.

The *cosecant* is that portion of the terminal line, or terminal line produced, included between the centre and the cotangent.

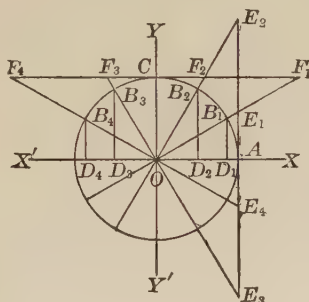
And with regard to algebraic signs,

Sines and tangents measured *above* XX' are *positive*, and *below*, *negative*; cosines and cotangents measured to the *right* of YY' are *positive*, and to the *left*, *negative*; secants and cosecants measured on the terminal line itself are *positive*, and on the terminal line *produced through O*, *negative*.

The above are called the *line values* of the trigonometric functions.

They simply *represent* the values of the functions when the radius is 1; that is, the *numerical value* of the sine of an angle is the same as the *number* which expresses the length of the perpendicular drawn to XX' from the intersection of the circumference and terminal line.

59. *To trace the changes in the sine, cosine, tangent, cotangent, secant, and cosecant of an angle as the angle increases from 0° to 360° .*



Let AB_4 be a circle whose radius is 1.

Let the terminal line start from the position OA , and revolve about point O as a pivot towards the position OC .

Then since the sine of the angle commences with the value 0, and assumes in succession the values B_1D_1 , B_2D_2 , OC , B_3D_3 , B_4D_4 , etc. (§ 58), it is evident that, as the angle increases from 0° to 90° , the sine increases from 0 to 1; from 90° to 180° , it decreases from 1 to 0; from 180° to 270° , it decreases (algebraically) from 0 to -1 ; and from 270° to 360° , it increases from -1 to 0.

Since the cosine commences with the value OA , and assumes in succession the values OD_1 , OD_2 , 0, $-OD_3$, $-OD_4$, etc., from 0° to 90° , it decreases from 1 to 0; from 90° to 180° , it decreases from 0 to -1 ; from 180° to 270° , it increases from -1 to 0; and from 270° to 360° , it increases from 0 to 1.

Since the tangent commences with the value 0, and assumes in succession the values AE_1 , AE_2 , ∞ (see Note to § 25), $-AE_3$, $-AE_4$, etc., from 0° to 90° , it increases from 0 to ∞ ; from 90° to 180° , it increases from $-\infty$ to 0; from 180° to 270° , it increases from 0 to ∞ ; and from 270° to 360° , it increases from $-\infty$ to 0.

Since the cotangent commences at ∞ , and assumes in succession the values $CF_1, CF_2, 0, -CF_3, -CF_4$, etc., from 0° to 90° , it decreases from ∞ to 0; from 90° to 180° , it decreases from 0 to $-\infty$; from 180° to 270° , it decreases from ∞ to 0; and from 270° to 360° , it decreases from 0 to $-\infty$.

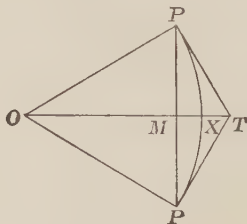
Since the secant commences with the value OA , and assumes in succession the values OE_1 , OE_2 , ∞ , $-OE_3$, $-OE_4$, etc., from 0° to 90° , it increases from 1 to ∞ ; from 90° to 180° , it increases from $-\infty$ to -1 ; from 180° to 270° , it decreases from -1 to $-\infty$; and from 270° to 360° , it decreases from ∞ to 1.

Since the cosecant commences at ∞ , and assumes in succession the values $OF_1, OF_2, OC, OF_3, OF_4$, etc., from 0° to 90° , it decreases from ∞ to 1; from 90° to 180° , it increases from 1 to ∞ ; from 180° to 270° , it increases from $-\infty$ to -1 ; and from 270° to 360° , it decreases from -1 to $-\infty$.

60. Limiting Values of $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$.

To find the limiting values of the fractions $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$ when x is indefinitely decreased.

Note We suppose x to be expressed in *circular measure* (§ 50).



Let $OPXP'$ be a sector of a circle; $\angle POP'$ being $< 180^\circ$.

Draw lines PT and $P'T$ tangent to the arc at P and P' , respectively; also, lines OT and PP' intersecting at M .

By Geometry, $PT = P'T$.

Then, OT bisects PP' at right angles, and also bisects arc PP' at X .

Let $\angle XOP = \angle XOP' = x$.

By Geometry, arc $PP' >$ chord PP' , and $< PTP'$.

Whence, arc $PX > PM$, and $< PT$.

Therefore, $\frac{\text{arc } PX}{OP} > \frac{PM}{OP}$, and $< \frac{PT}{OP}$.

Or by § 50, circ. meas. $x > \sin x$, and $< \tan x$.

Representing the circular measure of x by x simply, and dividing through by $\sin x$, we have

$$\frac{x}{\sin x} > 1, \text{ and } < \frac{\tan x}{\sin x} \text{ or } \frac{1}{\cos x} \text{ (§ 36).}$$

Whence, $\frac{\sin x}{x} < 1$, and $> \cos x$.

But when x is indefinitely decreased, $\cos x$ approaches the limit 1 (§ 22).

Hence, $\frac{\sin x}{x}$ approaches the limit 1 when x is indefinitely decreased.

Again, $\frac{\tan x}{x} = \frac{\sin x}{x \cos x} = \frac{\sin x}{x} \times \frac{1}{\cos x}$.

But $\frac{\sin x}{x}$ and $\frac{1}{\cos x}$ approach the limit 1 when x is indefinitely decreased.

Hence, $\frac{\tan x}{x}$ approaches the limit 1 when x is indefinitely decreased.

V. LOGARITHMS.

61. Every positive number may be expressed, exactly or approximately, as a power of 10.

Thus, $100 = 10^2$; $13 = 10^{1.1139\dots}$; etc.

When thus expressed, the corresponding exponent is called its **Logarithm to the Base 10**.

Thus, 2 is the logarithm of 100 to the base 10; a relation which is written $\log_{10} 100 = 2$, or simply $\log 100 = 2$.

62. Logarithms of numbers to the base 10 are called *Common Logarithms*, and, collectively, form the *Common System*.

They are the only ones used for numerical computations.

Any positive number, except unity, may be taken as the base of a system of logarithms; thus, if $a^x = m$, where a and m are positive numbers, then $x = \log_a m$.

Note. A negative number is not considered as having a logarithm.

63. We have by Algebra,

$$10^0 = 1,$$

$$10^{-1} = \frac{1}{10} = .1,$$

$$10^1 = 10,$$

$$10^{-2} = \frac{1}{10^2} = .01,$$

$$10^2 = 100,$$

$$10^{-3} = \frac{1}{10^3} = .001, \text{ etc.}$$

Whence by the definition of § 61,

$$\log 1 = 0,$$

$$\log .1 = -1 = 9 - 10,$$

$$\log 10 = 1,$$

$$\log .01 = -2 = 8 - 10,$$

$$\log 100 = 2,$$

$$\log .001 = -3 = 7 - 10, \text{ etc.}$$

Note. The second form for $\log .1$, $\log .01$, etc., is preferable in practice. If no base is expressed, the base 10 is understood.

64. It is evident from § 63 that the logarithm of a number greater than 1 is positive, and the logarithm of a number between 0 and 1 negative.

65. If a number is not an exact power of 10, its common logarithm can only be expressed approximately.

The integral part of the logarithm is called the *characteristic*, and the decimal part the *mantissa*.

For example, $\log 13 = 1.1139$.

Here, the characteristic is 1, and the mantissa .1139.

For reasons which will appear hereafter, only the mantissa of the logarithm is given in a table of logarithms of numbers; the characteristic must be found by aid of the rules of §§ 66 and 67.

66. It is evident from § 63 that the logarithm of a number between

1 and 10 is 0 + a decimal;

10 and 100 is 1 + a decimal;

100 and 1000 is 2 + a decimal; etc.

Therefore, the characteristic of the logarithm of a number with *one* place to the left of the decimal point, is 0; with *two* places to the left of the decimal point, is 1; with *three* places to the left of the decimal point, is 2; etc.

Hence, *the characteristic of the logarithm of a number greater than 1 is 1 less than the number of places to the left of the decimal point.*

For example, the characteristic of $\log 906328.5$ is 5.

67. In like manner, the logarithm of a number between

1 and .1 is 9 + a decimal - 10;

.1 and .01 is 8 + a decimal - 10;

.01 and .001 is 7 + a decimal - 10; etc.

Therefore, the characteristic of the logarithm of a decimal with *no* ciphers between the decimal point and first significant figure, is 9, with -10 after the mantissa; of a decimal with *one* cipher between the point and first significant figure is 8, with -10 after the mantissa; of a decimal with *two* ciphers between the point and first significant figure is 7, with -10 after the mantissa; etc.

Hence, to find the characteristic of the logarithm of a number between 0 and 1, subtract the number of ciphers between the decimal point and first significant figure from 9, writing -10 after the mantissa.

For example, the characteristic of $\log .007023$ is 7, with -10 written after the mantissa.

Note 1. It is customary in practice to omit the -10 after the mantissa of a negative logarithm; but it should be allowed for in the result. Beginners should always write it.

Note 2. Some writers combine the two portions of the characteristic, and write the result as a *negative characteristic* before the mantissa.

Thus, instead of $7.6036 - 10$, the student will frequently find $\bar{3}.6036$, a minus sign being written over the characteristic to denote that it alone is negative, the mantissa being always positive.

PROPERTIES OF LOGARITHMS.

68. In any system, the logarithm of 1 is 0.

For by Algebra, $a^0 = 1$; whence by § 62, $\log_a 1 = 0$.

69. In any system, the logarithm of the base is 1.

For $a^1 = a$; whence, $\log_a a = 1$.

70. In any system whose base is greater than 1, the logarithm of 0 is $-\infty$.

For if a is greater than 1, $a^{-\infty} = \frac{1}{a^{\infty}} = \frac{1}{\infty} = 0$.

Whence by § 62, $\log_a 0 = -\infty$.

Note. No literal meaning can be attached to such a result as $\log_a 0 = -\infty$.

It must be interpreted as follows :

If, in any system whose base is greater than unity, a number approaches the limit 0, its logarithm is negative, and increases without limit in absolute value.

71. *In any system, the logarithm of a product is equal to the sum of the logarithms of its factors.*

Assume the equations

$$\left. \begin{array}{l} a^x = m \\ a^y = n \end{array} \right\}; \text{ whence by § 62, } \left\{ \begin{array}{l} x = \log_a m, \\ y = \log_a n. \end{array} \right.$$

Multiplying the assumed equations,

$$a^x \times a^y = mn, \text{ or } a^{x+y} = mn.$$

$$\text{Whence, } \log_a mn = x + y = \log_a m + \log_a n.$$

In like manner, the theorem may be proved for the product of three or more factors.

72. By aid of § 71, the logarithm of a composite number may be found when the logarithms of its factors are known.

1. Given $\log 2 = .3010$ and $\log 3 = .4771$; find $\log 72$.

$$\begin{aligned} \log 72 &= \log(2 \times 2 \times 2 \times 3 \times 3) \\ &= \log 2 + \log 2 + \log 2 + \log 3 + \log 3 \text{ (§ 71)} \\ &= 3 \times \log 2 + 2 \times \log 3 = .9030 + .9542 = 1.8572. \end{aligned}$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 5 = .6990$, and $\log 7 = .8451$, find:

- | | | | |
|-----------------------|------------------------|-------------------------|---------------------------|
| 2. $\log 35$. | 6. $\log 147$. | 10. $\log 288$. | 14. $\log 2205$. |
| 3. $\log 30$. | 7. $\log 225$. | 11. $\log 686$. | 15. $\log 7875$. |
| 4. $\log 98$. | 8. $\log 175$. | 12. $\log 504$. | 16. $\log 5832$. |
| 5. $\log 84$. | 9. $\log 420$. | 13. $\log 375$. | 17. $\log 14112$. |

73. *In any system, the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.*

Assume the equations

$$\left. \begin{array}{l} a^x = m \\ a^y = n \end{array} \right\}; \text{ whence, } \begin{cases} x = \log_a m, \\ y = \log_a n. \end{cases}$$

Dividing the assumed equations,

$$\frac{a^x}{a^y} = \frac{m}{n}, \text{ or } a^{x-y} = \frac{m}{n}.$$

Whence, $\log_a \frac{m}{n} = x - y = \log_a m - \log_a n.$

74. 1. Given $\log 2 = .3010$; find $\log 5$.

$$\log 5 = \log \frac{10}{2} = \log 10 - \log 2 \text{ (§ 73)} = 1 - .3010 = .6990.$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, and $\log 7 = .8451$, find:

- | | | | |
|-------------------------|--------------------------|---------------------------|----------------------------|
| 2. $\log \frac{10}{7}.$ | 5. $\log 33\frac{1}{3}.$ | 8. $\log \frac{162}{49}.$ | 11. $\log 23\frac{5}{8}.$ |
| 3. $\log \frac{7}{4}.$ | 6. $\log \frac{27}{16}.$ | 9. $\log 4\frac{4}{9}.$ | 12. $\log \frac{196}{25}.$ |
| 4. $\log 45.$ | 7. $\log 105.$ | 10. $\log 525.$ | 13. $\log 96\frac{3}{7}.$ |

75. *In any system, the logarithm of any power of a quantity is equal to the logarithm of the quantity multiplied by the exponent of the power.*

Assume the equation $a^x = m$; whence, $x = \log_a m.$

Raising both members of the assumed equation to the p th power,

$$a^{px} = m^p; \text{ whence, } \log_a m^p = px = p \log_a m.$$

76. *In any system, the logarithm of any root of a quantity is equal to the logarithm of the quantity divided by the index of the root.*

For, $\log_a \sqrt[r]{m} = \log_a (m^{\frac{1}{r}}) = \frac{1}{r} \log_a m$ (§ 75).

77. 1. Given $\log 2 = .3010$; find $\log 2^{\frac{5}{3}}$.

$$\log 2^{\frac{5}{3}} = \frac{5}{3} \times \log 2 = \frac{5}{3} \times .3010 = .5017.$$

Note. To multiply a logarithm by a fraction, multiply first by the numerator, and divide the result by the denominator.

2. Given $\log 3 = .4771$; find $\log \sqrt[8]{3}$.

$$\log \sqrt[8]{3} = \frac{\log 3}{8} = \frac{.4771}{8} = .0596.$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, and $\log 7 = .8451$, find:

3. $\log 2^8$. 6. $\log 42^5$. 9. $\log \sqrt[5]{3}$. 12. $\log \sqrt[4]{28}$.

4. $\log 7^{\frac{5}{7}}$. 7. $\log 15^{\frac{2}{5}}$. 10. $\log \sqrt[6]{7}$. 13. $\log \sqrt[10]{324}$.

5. $\log 5^{\frac{3}{2}}$. 8. $\log 48^{\frac{4}{3}}$. 11. $\log \sqrt[9]{5}$. 14. $\log \sqrt[7]{735}$.

15. Find $\log (2^{\frac{1}{3}} \times 3^{\frac{5}{4}})$.

By § 71, $\log (2^{\frac{1}{3}} \times 3^{\frac{5}{4}}) = \log 2^{\frac{1}{3}} + \log 3^{\frac{5}{4}} = \frac{1}{3} \log 2 + \frac{5}{4} \log 3$
 $= .1003 + .5964 = .6967.$

Find the values of the following:

16. $\log \sqrt[9]{7}$. 18. $\log \frac{\sqrt[4]{7}}{\sqrt[5]{2}}$. 20. $\log \frac{\sqrt[3]{10}}{\sqrt[7]{2}}$. 22. $\log \left(\frac{14}{5}\right)^{\frac{1}{4}}$.

17. $\log 5^{\frac{11}{2}}$. 19. $\log \frac{3^{\frac{5}{2}}}{5^{\frac{2}{3}}}$. 21. $\log \frac{3^{\frac{5}{3}}}{\sqrt[7]{7}}$. 23. $\log (2^{\frac{2}{3}} \times 21^{\frac{1}{2}})$.

78. *To prove the relation*

$$\log_b m = \frac{\log_a m}{\log_a b}.$$

Assume the equations

$$\left. \begin{array}{l} a^x = m \\ b^y = m \end{array} \right\}; \text{ whence, } \left\{ \begin{array}{l} x = \log_a m, \\ y = \log_b m. \end{array} \right.$$

From the assumed equations,

$$a^x = b^y.$$

Taking the y th root of both members,

$$a^{\frac{x}{y}} = b.$$

Therefore, $\log_a b = \frac{x}{y}$, or $y = \frac{x}{\log_a b}$.

That is, $\log_b m = \frac{\log_a m}{\log_a b}$.

79. *To prove the relation*

$$\log_b a \times \log_a b = 1.$$

Putting $m = a$ in the result of § 78, we have

$$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b} \text{ (§ 69).}$$

Whence, $\log_b a \times \log_a b = 1$.

80. *In the common system, the mantissæ of the logarithms of numbers having the same sequence of figures are equal.*

Suppose, for example, that $\log 3.053 = .4847$.

Then, $\log 305.3 = \log (100 \times 3.053) = \log 100 + \log 3.053$
 $= 2 + .4847 = 2.4847$;

$\log .03053 = \log (.01 \times 3.053) = \log .01 + \log 3.053$
 $= 8 - 10 + .4847 = 8.4847 - 10$; etc.

It is evident from the above that, if a number be multiplied or divided by any integral power of 10, producing another number with the same sequence of figures, the mantissæ of their logarithms will be equal.

The reason will now be seen for the statement made in § 65, that only the mantissæ are given in a table of logarithms of numbers.

For, to find the logarithm of any number, we have only to take from the table the mantissa corresponding to its sequence of figures, and the characteristic may then be prefixed in accordance with the rules of §§ 66 or 67.

Thus, if $\log 3.053 = .4847$, then

$$\begin{array}{ll} \log 30.53 = 1.4847, & \log .3053 = 9.4847 - 10, \\ \log 305.3 = 2.4847, & \log .03053 = 8.4847 - 10, \\ \log 3053. = 3.4847, & \log .003053 = 7.4847 - 10, \text{ etc.} \end{array}$$

This property is enjoyed only by the common system of logarithms, and constitutes its superiority over others for the purposes of numerical computation.

81. 1. Given $\log 2 = .3010$, $\log 3 = .4771$; find $\log .00432$.

We have $\log 432 = \log (2^4 \times 3^3) = 4 \log 2 + 3 \log 3 = 2.6353$.

Then by § 80, the *mantissa* of the result is .6353.

Whence by § 67, $\log .00432 = 7.6353 - 10$.

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, and $\log 7 = .8451$, find:

- | | | |
|-------------------|---------------------|-----------------------------------|
| 2. $\log 2.4$. | 6. $\log .00135$. | 10. $\log .1029$. |
| 3. $\log 16.8$. | 7. $\log 5880$. | 11. $\log 201.6$. |
| 4. $\log .81$. | 8. $\log .0245$. | 12. $\log \sqrt[6]{7.5}$. |
| 5. $\log .0192$. | 9. $\log .000486$. | 13. $\log (12.6)^{\frac{3}{2}}$. |

USE OF THE TABLE OF LOGARITHMS OF NUMBERS.

(For directions as to the use of the Table of Logarithms of Numbers, see pages 1 to 4 of the Introduction to the author's New Four Place Logarithmic Tables.)

EXAMPLES.

82. Find the logarithms of the following numbers :

1. 80.	6. .03294.	11. .0007178.
2. 6.3.	7. .5205.	12. 5.1809.
3. .298.	8. 20.08.	13. 1036.5.
4. 772.3.	9. 92461.	14. .086676.
5. 1056.	10. .0040322.	15. .000011507.

Find the numbers corresponding to the following logarithms:

16. 1.8055.	21. 8.1646 — 10.	26. 1.6482.
17. 9.4487 — 10.	22. 7.5209 — 10.	27. 6.0450 — 10.
18. 0.2165.	23. 2.0095.	28. 4.8016.
19. 3.9487.	24. 0.9774.	29. 8.1142 — 10.
20. 2.7371.	25. 9.3178 — 10.	30. 5.7015 — 10.

APPLICATIONS.

83. The approximate value of an arithmetical quantity, in which the operations indicated involve only multiplication, division, involution, or evolution, may be conveniently found by logarithms.

The utility of the process consists in the fact that addition takes the place of multiplication, subtraction of division, multiplication of involution, and division of evolution.

Note. In computations with four-place logarithms, the results cannot usually be depended upon to more than *four* significant figures.

84. 1. Find the value of $.0631 \times 7.208 \times .51272$.

By § 71, $\log (.0631 \times 7.208 \times .51272)$
 $= \log .0631 + \log 7.208 + \log .51272.$

$$\log .0631 = 8.8000 - 10$$

$$\log 7.208 = 0.8578$$

$$\log .51272 = 9.7099 - 10$$

Adding, $\log \text{ of result} = 19.3677 - 20$
 $= 9.3677 - 10. \quad (\text{See Note 1.})$

Number corresponding to $9.3677 - 10 = .2332$.

Note 1. If the sum is a negative logarithm, it should be written in such a form that the negative portion of the characteristic may be -10 .

Thus, $19.3677 - 20$ is written in the form $9.3677 - 10$.

2. Find the value of $\frac{336.8}{7984}$.

By § 73, $\log \frac{336.8}{7984} = \log 336.8 - \log 7984.$

$$\log 336.8 = 12.5273 - 10 \quad (\text{See Note 2.})$$

$$\log 7984 = 3.9022$$

Subtracting, $\log \text{ of result} = 8.6251 - 10$

Number corresponding $= .04218$.

Note 2. To subtract a greater logarithm from a less, or to subtract a negative logarithm from a positive, increase the characteristic of the minuend by 10, writing -10 after the mantissa to compensate.

Thus, to subtract 3.9022 from 2.5273, write the minuend in the form $12.5273 - 10$; subtracting 3.9022 from this, the result is $8.6251 - 10$.

3. Find the value of $(.07396)^5$.

By § 75, $\log (.07396)^5 = 5 \times \log .07396.$

$$\log .07396 = 8.8690 - 10$$

5

$$44.3450 - 50$$

$$= 4.3450 - 10 \quad (\text{See Note 1.})$$

$$= \log .000002213.$$

4. Find the value of $\sqrt[3]{.035063}$.

By § 76, $\log \sqrt[3]{.035063} = \frac{1}{3} \log .035063$.

$$\begin{array}{r} \log .035063 = 8.5449 - 10 \\ \quad \quad \quad 20. \quad \quad - 20 \quad (\text{See Note 3.}) \\ \hline 3) 28.5449 - 30 \\ \hline 9.5150 - 10 = \log .3274. \end{array}$$

Note 3. To divide a negative logarithm, write it in such a form that the negative portion of the characteristic may be exactly divisible by the divisor, with -10 as the quotient.

Thus, to divide $8.5449 - 10$ by 3, we write the logarithm in the form $28.5449 - 30$; dividing this by 3, the quotient is $9.5150 - 10$.

85. Arithmetical Complement.

The *Arithmetical Complement* of the logarithm of a number, or, briefly, the *Cologarithm* of the number, is the logarithm of the reciprocal of that number.

Thus, $\text{colog } 409 = \log \frac{1}{409} = \log 1 - \log 409$.

$$\begin{array}{r} \log 1 = 10. \quad \quad - 10 \quad (\text{Note 2, § 84.}) \\ \log 409 = 2.6117 \\ \hline \therefore \text{colog } 409 = 7.3883 - 10. \end{array}$$

Again, $\text{colog } .067 = \log \frac{1}{.067} = \log 1 - \log .067$.

$$\begin{array}{r} \log 1 = 10. \quad \quad - 10 \\ \log .067 = 8.8261 - 10 \\ \hline \therefore \text{colog } .067 = 1.1739. \end{array}$$

It follows from the above that *the cologarithm of a number may be found by subtracting its logarithm from $10 - 10$* .

Note. The cologarithm may be obtained by subtracting the last *significant* figure of the logarithm from 10, and each of the others from 9, -10 being written after the result in the case of a positive logarithm.

86. Example. Find the value of $\frac{.51384}{8.709 \times .0946}$.

$$\begin{aligned}\log \frac{.51384}{8.709 \times .0946} &= \log \left(.51384 \times \frac{1}{8.709} \times \frac{1}{.0946} \right) \\ &= \log .51384 + \log \frac{1}{8.709} + \log \frac{1}{.0946} \\ &= \log .51384 + \text{colog } 8.709 + \text{colog } .0946. \\ \log .51384 &= 9.7109 - 10 \\ \text{colog } 8.709 &= 9.0601 - 10 \\ \text{colog } .0946 &= \frac{1.0241}{9.7951 - 10} = \log .6239.\end{aligned}$$

It is evident from the above example that the logarithm of a fraction either of whose terms is the product of factors, may be found by the following rule:

Add together the logarithms of the factors of the numerator, and the cologarithms of the factors of the denominator.

EXAMPLES.

Note. A *negative* number has no common logarithm (§ 62, Note).

If such numbers occur in computation, they should be treated as if they were positive, and the *sign* of the result determined irrespective of the logarithmic work.

Thus, in Ex. 3, § 87, the value of $439.2 \times (-7.1367)$ is obtained by finding the value of 439.2×7.1367 , and putting a negative sign before the result. See also Ex. 33.

87. Find by logarithms the values of the following:

- | | |
|--|---|
| <p>✓ 1. $3.145 \times .6839$.</p> <p>2. $847.6 \times .02287$.</p> <p>3. $439.2 \times (-7.1367)$.</p> <p>7. $\frac{486.7}{76.52}$.</p> <p>8. $\frac{1.0548}{34.96}$.</p> | <p>4. $(-9.0654) \times (-.010785)$.</p> <p>✓ 5. $.36552 \times .025208$.</p> <p>6. $-.0019036 \times 57.143$.</p> <p>9. $\frac{-.2709}{.08683}$.</p> <p>11. $\frac{8062.4}{9.5073}$.</p> <p>12. $\frac{.0001798}{-.033166}$.</p> |
|--|---|

13. $\frac{38.961 \times .695}{4994 \times .0045}$ 15. $\frac{(-.87028) \times 37}{(-.0659) \times (-42.32)}$
14. $\frac{715 \times (-.02416)}{(-.516) \times 142.07}$ 16. $\frac{.08214 \times (-73.4)}{.84 \times 2808.7}$
17. $(7.795)^4$ 22. $(.095129)^{\frac{5}{2}}$ 27. $\sqrt[8]{100}$
18. $(.8328)^7$ 23. $(.00010594)^{\frac{5}{3}}$ 28. $\sqrt[4]{.1995}$
19. $(-25.144)^3$ 24. $\sqrt{5}$ 29. $\sqrt[6]{.072563}$
20. $(.01)^{\frac{3}{4}}$ 25. $\sqrt[5]{2}$ 30. $\sqrt[3]{.0026139}$
21. $(-964.8)^{\frac{4}{5}}$ 26. $\sqrt[9]{-6}$ 31. $\sqrt[7]{-.00095174}$

32. Find the value of $\frac{2\sqrt[3]{5}}{3^{\frac{5}{6}}}$.

By § 86,

$$\begin{aligned}\log \frac{2\sqrt[3]{5}}{3^{\frac{5}{6}}} &= \log 2 + \log \sqrt[3]{5} + \text{colog } 3^{\frac{5}{6}} \\ &= \log 2 + \frac{1}{3} \log 5 + \frac{5}{6} \text{colog } 3.\end{aligned}$$

$$\log 2 = 0.3010$$

$$\log 5 = 0.6990;$$

$$\text{divide by } 3 = 0.2330$$

$$\text{colog } 3 = 9.5229 - 10; \text{ multiply by } \frac{5}{6} = 9.6024 - 10$$

$$0.1364 = \log 1.369.$$

33. Find the value of $\sqrt[3]{\frac{-.03296}{7.962}}$.

$$\log \sqrt[3]{\frac{-.03296}{7.962}} = \frac{1}{3} \log \frac{.03296}{7.962} = \frac{1}{3} (\log .03296 - \log 7.962).$$

$$\log .03296 = 8.5180 - 10$$

$$\log 7.962 = 0.9010$$

$$\begin{array}{r} 3 \overline{) 27.6170 - 30} \end{array}$$

$$9.2057 - 10 = \log .1606.$$

Result, $-.1606$.

Find the values of the following:

$$34. 4^{\frac{4}{3}} \times 7^{\frac{2}{5}}. \quad 39. \left(-\frac{4400}{6928}\right)^{\frac{1}{5}}. \quad 44. \sqrt[3]{3} \times \sqrt[5]{5} \times \sqrt[7]{7}.$$

$$35. \frac{3^{\frac{5}{8}}}{8^{\frac{2}{5}}}. \quad 40. \sqrt{\frac{276.9}{940}}. \quad 45. \left(\frac{76.1 \times .0593}{1.307}\right)^{\frac{3}{4}}.$$

$$36. \sqrt[10]{\frac{79}{46}}. \quad 41. \frac{5^{\frac{7}{4}}}{\sqrt[3]{-.1}}. \quad 46. \sqrt[3]{-\frac{75.44}{31.4 \times .415}}.$$

$$37. \frac{(.001)^{\frac{3}{5}}}{\sqrt[5]{7}}. \quad 42. \frac{-\sqrt[4]{1000}}{(-.6)^{\frac{4}{3}}}. \quad 47. \frac{\sqrt[4]{.0009657}}{\sqrt[3]{.0049784}}.$$

$$38. \frac{\sqrt{.08}}{(-10)^{\frac{8}{5}}}. \quad 43. \sqrt[6]{\frac{3}{5}} \div \sqrt[5]{\frac{7}{8}}. \quad 48. \frac{-(.25693)^{\frac{6}{5}}}{(-.8346)^{\frac{7}{8}}}.$$

$$49. (25.467)^{10} \times (-.052)^{12}. \quad 54. \frac{\sqrt[3]{-.7664} \times 1.2809}{(.00259)^{\frac{3}{2}}}.$$

$$50. \sqrt[8]{5106.5} \times .00003109.$$

$$51. (837.5 \times .0094325)^{\frac{2}{7}}. \quad 55. \frac{\sqrt[4]{.05287}}{\sqrt{.374} \times \sqrt[9]{.0078359}}.$$

$$52. (4.8672)^{\frac{7}{2}} \times (.17544)^{\frac{1}{5}}.$$

$$53. \frac{\sqrt{3.929} \times \sqrt[4]{65.48}}{\sqrt[6]{721.33}}. \quad 56. \frac{\sqrt[5]{.04142} \times (-.947^{\frac{3}{4}})}{38.014}.$$

$$57. .083184 \times (.2682)^3 \times (56.1)^{\frac{5}{2}}.$$

$$58. \frac{.0005616 \times \sqrt[7]{424.65}}{(6.73)^4 \times (.03194)^{\frac{5}{8}}}.$$

$$59. \frac{485.7 \times (.07301)^7 \times \sqrt[8]{35.6}}{(9.1273)^6 \times (.7095)^{\frac{3}{5}}}.$$

$$60. \sqrt[3]{\left[\frac{(-.95048)^5 \times (8473)^{\frac{4}{3}}}{(-2080.9) \times \sqrt[6]{.0572}}\right]}.$$

$$61. \frac{\sqrt[5]{-.003012} \times 1.955}{(-.843)^8 \times \sqrt[4]{17959} \times (-560.6)^{\frac{7}{2}}}.$$

EXAMPLES IN THE USE OF TRIGONOMETRIC TABLES.

(For directions, see pages 4 to 8 of the Introduction to the author's New Four Place Logarithmic Tables.)

88. Tables of Logarithmic Sines, Cosines, etc.

Find the values of the following :

- | | |
|-------------------------------|------------------------------------|
| 1. $\log \tan 35^\circ 39'$. | 6. $\log \sin 30^\circ 37.2'$. |
| 2. $\log \sin 61^\circ 58'$. | 7. $\log \cos 55^\circ 21' 48''$. |
| 3. $\log \cot 12^\circ 34'$. | 8. $\log \cot 48^\circ 3' 43''$. |
| 4. $\log \cos 26^\circ 56'$. | 9. $\log \sec 80^\circ 7'$. |
| 5. $\log \tan 82^\circ 3'$. | 10. $\log \csc 65^\circ 12'$. |

Find the angles corresponding in the following :

- | | |
|---------------------------------|---------------------------------|
| 11. $\log \tan = 0.9164$. | 16. $\log \cot = 0.2154$. |
| 12. $\log \cos = 9.9221 - 10$. | 17. $\log \sin = 9.1891 - 10$. |
| 13. $\log \sin = 9.8619 - 10$. | 18. $\log \tan = 8.9668 - 10$. |
| 14. $\log \cot = 9.4700 - 10$. | 19. $\log \csc = 0.1888$. |
| 15. $\log \cos = 9.2204 - 10$. | 20. $\log \sec = 0.4032$. |

Tables of Natural Sines, Cosines, etc.

Find the values of the following :

- | | |
|---------------------------|---------------------------|
| 21. $\sin 17^\circ 13'$. | 23. $\tan 35^\circ 7'$. |
| 22. $\cos 75^\circ 38'$. | 24. $\cot 68^\circ 46'$. |

Find the angles corresponding in the following :

- | | |
|----------------------|-----------------------|
| 25. $\sin = .7385$. | 27. $\tan = 1.1897$. |
| 26. $\cos = .9280$. | 28. $\cot = 1.8207$. |

VI. SOLUTION OF RIGHT TRIANGLES.

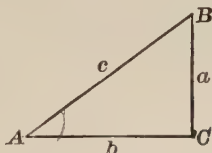
89. The *elements* of a triangle are its three sides and its three angles.

We know by Geometry that a triangle is, in general, completely determined when three of its elements are known, provided one of them is a side.

The *solution* of a triangle is the process of computing the unknown from the given elements.

90. To solve a *right triangle*, two elements must be given in addition to the right angle, one of which must be a side.

The various cases which can occur may all be solved by aid of the following formulæ:



$$\sin A = \frac{a}{c}.$$

$$\cos A = \frac{b}{c}.$$

$$\tan A = \frac{a}{b}.$$

$$\sin B = \frac{b}{c}.$$

$$\cos B = \frac{a}{c}.$$

$$\tan B = \frac{b}{a}.$$

91. CASE I. When the given elements are a side and an angle.

The formula for computing either of the remaining sides may be found by the following rule:

Take that function of the angle which involves the given side and the required side.

1. Given $c = 23$, $B = 21^{\circ} 33'$. Find a and b .

In this case, the formulæ to be used are

$$\cos B = \frac{a}{c}, \text{ and } \sin B = \frac{b}{c}.$$

Whence,

$$a = c \cos B, \text{ and } b = c \sin B.$$

(A)

Solution by Natural Functions.

$$a = 23 \times \cos 21^\circ 33' = 23 \times .9301 = 21.39.$$

$$b = 23 \times \sin 21^\circ 33' = 23 \times .3673 = 8.448.$$

Solution by Logarithms.

Taking the logarithms of both members, in formulæ (A),

$$\log a = \log c + \log \cos B, \text{ and } \log b = \log c + \log \sin B.$$

$\log c = 1.3617$	$\log c = 1.3617$
$\log \cos B = \underline{9.9685 - 10}$	$\log \sin B = \underline{9.5651 - 10}$
$\log a = 1.3302$	$\log b = 0.9268$
$a = 21.39.$	$b = 8.448.$

Note. In examples under Case I. in which the given sides are numbers of not more than two significant figures, and the operations indicated involve only multiplication, it is usually shorter to employ Natural Functions.

In such a case, the results cannot be depended upon to more than *four* significant figures.

2. Given $a = .2359$, $A = 67^\circ 18'$. Find b and c .

In this case, $\tan A = \frac{a}{b}$, and $\sin A = \frac{a}{c}$.

Whence, $b = \frac{a}{\tan A}$, and $c = \frac{a}{\sin A}$.

By logarithms,

$$\log b = \log a - \log \tan A, \text{ and } \log c = \log a - \log \sin A.$$

$\log a = 9.3727 - 10$	$\log a = 9.3727 - 10$
$\log \tan A = \underline{0.3785}$	$\log \sin A = \underline{9.9650 - 10}$
$\log b = 8.9942 - 10$	$\log c = 9.4077 - 10$
$b = .09868.$	$c = .2557.$

92. CASE II. When both given elements are sides.

First calculate one of the angles by aid of either formula involving the given elements, and then compute the remaining side by the rule of Case I.

Ex. Given $b = .1512$, $c = .3081$. Find A and a .

We first find A by the formula $\cos A = \frac{b}{c}$, and then a by the formula $\sin A = \frac{a}{c}$, or $a = c \sin A$.

By logarithms,

$$\log \cos A = \log b - \log c, \text{ and } \log a = \log c + \log \sin A.$$

$$\log b = 9.1796 - 10$$

$$\log c = 9.4887 - 10$$

$$\log c = 9.4887 - 10$$

$$\log \sin A = 9.9401 - 10$$

$$\log \cos A = 9.6909 - 10$$

$$\log a = 9.4288 - 10$$

$$A = 60^\circ 36.4'.$$

$$a = .2684.$$

93. In the Trigonometric solution of an example under Case II., it is necessary to find first one of the angles, and the remaining side may then be calculated.

But it is possible to compute the third side directly, without first finding the angle, by Geometry.

Thus, in the example of § 92, we have

$$a^2 + b^2 = c^2.$$

$$\text{Whence, } a = \sqrt{c^2 - b^2} = \sqrt{(c + b)(c - b)}.$$

$$\text{By logarithms, } \log a = \frac{1}{2} [\log (c + b) + \log (c - b)].$$

$$c + b = .4593; \log = 9.6621 - 10$$

$$c - b = .1569; \log = 9.1956 - 10$$

$$2 \overline{) 18.8577 - 20}$$

$$\log a = 9.4289 - 10$$

$$a = .2685.$$

If the given sides are a and b , the expression for c is $\sqrt{a^2 + b^2}$, which is not adapted to logarithmic computation.

In such a case, it is usually shorter to proceed as in § 92.

EXAMPLES.

94. Solve the following right triangles:

$$1. \text{ Given } A = 15^\circ, c = 7. \quad 3. \text{ Given } B = 50^\circ, b = 20.$$

$$2. \text{ Given } B = 68^\circ, a = 5. \quad 4. \text{ Given } a = .35, c = .62.$$

5. Given $a = 27$, $b = 42$. 8. Given $b = 586$, $c = 763$
6. Given $A = 38^\circ$, $a = 8.09$. 9. Given $A = 9^\circ$, $b = 937$.
7. Given $B = 65^\circ$, $c = .014$. 10. Given $a = 3.41$, $b = 2.87$
11. Given $A = 31^\circ 50'$, $a = 48.04$.
12. Given $A = 46^\circ 15'$, $c = 5280$.
13. Given $b = .0469$, $c = .0515$.
14. Given $B = 79^\circ 28'$, $b = 842$.
15. Given $B = 67^\circ 47'$, $c = .00954$.
16. Given $A = 43^\circ 30'$, $b = 26185$.
17. Given $a = 3402$, $b = 2317$.
18. Given $B = 82^\circ 6'$, $a = .08937$.
19. Given $b = 578.9$, $c = 2492$.
20. Given $A = 26^\circ 12'$, $c = .4694$.
21. Given $B = 14^\circ 53'$, $b = 1353$.
22. Given $B = 43^\circ 24'$, $a = .89658$.
23. Given $a = 99.46$, $c = 156.8$.
24. Given $A = 62^\circ 44'$, $b = 4.2492$.
25. Given $A = 74^\circ 17'$, $a = .000020386$.
26. Given $B = 29^\circ 56'$, $c = .00078144$.
27. Given $a = 63827$, $c = 92275$.
28. Given $A = 58^\circ 39'$, $c = 35.733$.
29. Given $B = 35^\circ 8'$, $b = 17269$.
30. Given $a = .0067239$, $b = .0038453$.

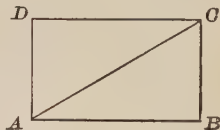
Solve the following isosceles triangles, in which A and B are the equal angles, and a , b , and c the sides opposite angles A , B , and C , respectively:

31. Given $A = 71^\circ$, $b = 39$.
32. Given $B = 36^\circ 40'$, $c = .4688$.

33. Given $C = 83^\circ 52'$, $b = 710.6$.
 34. Given $a = 6875$, $c = 11318$.
 35. Given $B = 29^\circ 7'$, $a = 2.569$.
 36. Given $A = 54^\circ 39'$, $c = 1.7255$.
 37. Given $C = 135^\circ 26'$, $c = .06377$.

MISCELLANEOUS PROBLEMS.

95. If AC is the diagonal of rectangle $ABCD$, and the side AB is horizontal and BC vertical, $\angle BAC$ is called the *angle of elevation* of point C from point A , and $\angle ACD$ the *angle of depression* of point A from point C .

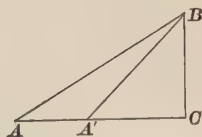


96. 1. From the top of a lighthouse, 150 feet above the sea, the angles of depression of two boats, in line with the lighthouse, are observed to be 12° and 30° , respectively. Find the distance between the boats.

Let A be the position of the first boat, A' of the second, B the top of the lighthouse, and C its foot.

Then,

$$\begin{aligned}
 AA' &= AC - A'C = BC \cot A - BC \cot BA'C \\
 &= 150 (\cot 12^\circ - \cot 30^\circ) \\
 &= 150 (4.7046 - 1.7321) \\
 &= 150 \times 2.9725 = 445.9 \text{ ft.}
 \end{aligned}$$



2. From the top of a lighthouse, 250 feet above the sea, the angle of depression of a buoy is observed to be 31° . Find the horizontal distance of the buoy.

3. If the radius of a circle is 834, what is the length of a chord which subtends an arc of 46° ?

4. A regular hexagon is circumscribed about a circle whose diameter is 59. Find the length of its side.

5. Find the angle of elevation of a road which rises a distance of 349 feet in a horizontal distance of five-eighths of a mile.

6. A regular polygon of nine sides is inscribed in a circle whose diameter is 68. Find the length of its side.

7. How far from the foot of a tower 121 feet in height must an observer stand so that the angle of elevation of its top may be 23° ?

8. A regular polygon, whose side is 7.6 and angle 144° , is circumscribed about a circle. Find the radius of the circle.

9. Find the angle of elevation of the sun when a monument whose height is 214.8 feet casts a shadow 167.4 feet in length.

10. Find the length of the diagonal of a regular pentagon whose side is 9.437.

11. At a distance of 41.6 feet from the base of a tower, the angle of elevation of its top is observed to be $59^\circ 36'$. Find its height.

12. The middle point of a chord of a circle, 24 units in length, is distant 7 units from the middle point of its subtended arc. How many degrees and minutes are there in the arc?

13. If the diameter of a circle is 6374, find the angle at the centre subtended by an arc whose chord is 2138.

14. If the radius of a circle is 9.54, and the distance from the middle point of a chord to the middle point of its subtended arc is 3.87, how many degrees and minutes are there in the arc?

15. From the top of a tower, the angle of depression of the extremity of a horizontal base line, 236.1 feet in length measured from the foot of the tower, is observed to be $29^\circ 48'$. Find the height of the tower.

16. A chord of a circle, whose length is 14.95, subtends an arc of $135^{\circ} 52'$. What is the distance from the middle point of the chord to the middle point of the arc?

17. If a vertical pole casts a shadow which is three-fourths its own length, what is the angle of elevation of the sun?

18. The radius of the inscribed circle of an equilateral triangle is .307. Find its perimeter, and the diameter of the circumscribed circle.

19. The side of a regular octagon is 23.68. Find the radii of its inscribed and circumscribed circles.

20. A chord of a circle subtends an arc of $70^{\circ} 24'$. If the length of the chord is 853.4, find the radius of the circle.

21. At a point 250 feet from the foot of a cliff surmounted by a lighthouse, the angle of elevation of the top of the lighthouse is 50° , and of its foot 30° . Find the height of the cliff, and of the lighthouse.

22. From the top of a cliff 378 feet above the sea, the angles of depression of two boats, in line with the observer, are observed to be $11^{\circ} 50'$ and $29^{\circ} 20'$, respectively. Find the distance between the boats.

23. At a distance of 169 feet from the foot of a tower surmounted by a pole, the tower subtends an angle of 35° , and the pole an angle of 12° . Find the length of the pole.

24. How many degrees and minutes are there in the arc included between two parallel chords, on the same side of the centre of a circle, whose distances from the centre are 5 and 7, respectively, the radius of the circle being 11?

25. From the top of a tower, the angle of depression of a stake is $31^{\circ} 29'$. What will be the angle of depression of the stake from a point half way to the top?

26. The diagonal of a regular pentagon is 43.92. Find the radius of its inscribed circle.

27. A railway runs from A to B , a horizontal distance of 1250 feet, at an angle of elevation of $8^{\circ} 12'$, and then from B to C , a horizontal distance of 375 feet, at an angle of elevation of $7^{\circ} 26'$. How many feet is C above the plane of A ?

28. From a point 200 feet from the foot of a tower surmounted by a pole, the angle of elevation of the top of the pole is 38° ; from a point 150 feet further, the angle of elevation of the foot of the pole is 22° . Find the height of the pole.

29. If the radius of the earth is 3956 miles, find the radius in miles of the arctic circle, latitude $66^{\circ} 32' \text{ N.}$

30. If the diameter of the earth is 7912 miles, what is the distance of the remotest point of the surface visible from the top of a mountain, $1\frac{1}{2}$ miles above the sea?

31. A flagpole 23 feet long surmounts a tower whose height is 98 feet. What angle does the flagpole subtend at a point on the ground 315 feet from the base of the tower?

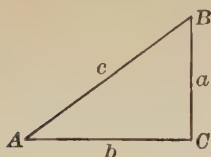
32. An observer notes that a spire bears due north from him, the angle of elevation of its top being $22^{\circ} 17'$. On going due east 550 feet, the spire bears 49° west of north. What is the height of the spire?

33. A regular pyramid stands on a square base, whose side is 50 feet. Each side of the base makes an angle of 69° with the lateral edge. Find the altitude of the pyramid.

34. A vessel is sailing due north at a uniform rate of speed. At 7.30 A.M., a lighthouse is observed to bear 70° west of north; at 8 A.M. it is due west, at a distance of 12 miles. Find the distance and bearing of the lighthouse at 9.30 A.M.

FORMULÆ FOR THE AREA OF A RIGHT TRIANGLE

97. CASE I. *Given the hypotenuse and an acute angle.*



Denoting the area by K , we have by Geometry,

$$2K = ab.$$

But by § 5, $a = c \sin A$, and $b = c \cos A$.

Whence, $2K = c^2 \sin A \cos A = \frac{1}{2} c^2 \sin 2A$, by (22).

$$\text{Then,} \quad 4K = c^2 \sin 2A. \quad (32)$$

$$\text{In like manner,} \quad 4K = c^2 \sin 2B. \quad (33)$$

CASE II. *Given an angle and its opposite side.*

$$\text{By § 2,} \quad b = a \cot A.$$

$$\text{Whence,} \quad 2K = a \times a \cot A = a^2 \cot A. \quad (34)$$

$$\text{In like manner,} \quad 2K = b^2 \cot B. \quad (35)$$

CASE III. *Given an angle and its adjacent side.*

$$\text{By § 2,} \quad b = a \tan B.$$

$$\text{Whence,} \quad 2K = a \times a \tan B = a^2 \tan B. \quad (36)$$

$$\text{In like manner,} \quad 2K = b^2 \tan A. \quad (37)$$

CASE IV. *Given the hypotenuse and another side.*

Since $a^2 + b^2 = c^2$, we have

$$2K = ab = a\sqrt{c^2 - a^2} = a\sqrt{(c+a)(c-a)}. \quad (38)$$

$$\text{In like manner,} \quad 2K = b\sqrt{(c+b)(c-b)}. \quad (39)$$

CASE V. *Given the two sides about the right angle.*

$$\text{In this case,} \quad 2K = ab. \quad (40)$$

EXAMPLES.

98. 1. Given $c = 10.36$, $B = 75^\circ$; find the area,

By (33), $4K = c^2 \sin 2B$.

Whence, $\log(4K) = 2 \log c + \log \sin 2B$.

$\log c = 1.0153$; multiply by 2 = 2.0306

$2B = 150^\circ$; $\log \sin = 9.6990 - 10$

$\log(4K) = 1.7296$

$\therefore 4K = 53.65$, and $K = 13.41$.

Note. To find $\log \sin 150^\circ$, take either $\log \cos 60^\circ$ or $\log \sin 30^\circ$.
(See page 7 of the Introduction to the author's New Four Place Logarithmic Tables.)

Find the areas of the following right triangles:

2. Given $A = 46^\circ$, $a = 2.717$.

3. Given $B = 35^\circ 16'$, $a = .557$.

4. Given $a = 283.17$, $b = 94.93$.

5. Given $b = 4.564$, $c = 7.176$.

6. Given $A = 53^\circ 9'$, $c = 13.84$.

7. Given $A = 20^\circ 57'$, $b = .05027$.

8. Given $a = .0861$, $c = .4806$.

9. Given $B = 67^\circ 48'$, $c = 67.409$.

10. Given $B = 75^\circ 34'$, $b = .0032056$.

11. Given $A = 81^\circ 23'$, $c = 195.84$.

VII. GENERAL PROPERTIES OF TRIANGLES.

99. *In any triangle, the sides are proportional to the sines of their opposite angles.*

I. To prove $a : b = \sin A : \sin B$. (41)

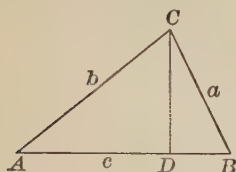


FIG. 1.

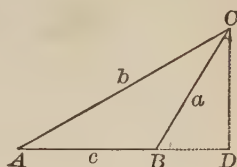


FIG. 2.

There will be two cases, according as angles A and B are both acute (Fig. 1), or one of them obtuse (Fig. 2).

In each case, draw line CD perpendicular to AB .

Then in each figure, $CD = b \sin A$ (§ 5).

Also in Fig. 1, $CD = a \sin B$.

And in Fig. 2, $CD = a \sin CBD$

$$= a \sin (180^\circ - B) = a \sin B \quad (\S\ 32).$$

Then in either case, $b \sin A = a \sin B$.

Whence by the theory of proportion,

$$a : b = \sin A : \sin B.$$

In like manner, $b : c = \sin B : \sin C$, (42)

and $c : a = \sin C : \sin A$. (43)

100. *In any triangle, the sum of any two sides is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

By (41). $a : b = \sin A : \sin B$.

Whence by composition and division,

$$a + b : a - b = \sin A + \sin B : \sin A - \sin B.$$

$$\text{Or,} \quad \frac{a + b}{a - b} = \frac{\sin A + \sin B}{\sin A - \sin B}.$$

$$\text{But,} \quad \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}, \text{ by (21).}$$

$$\text{Whence,} \quad \frac{a + b}{a - b} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}. \quad (44)$$

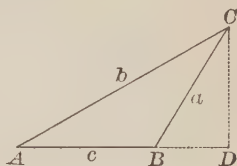
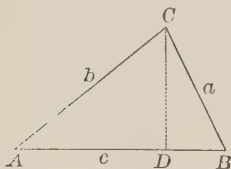
$$\text{In like manner,} \quad \frac{b + c}{b - c} = \frac{\tan \frac{1}{2}(B + C)}{\tan \frac{1}{2}(B - C)}, \quad (45)$$

$$\text{and} \quad \frac{c + a}{c - a} = \frac{\tan \frac{1}{2}(C + A)}{\tan \frac{1}{2}(C - A)}. \quad (46)$$

101. *In any triangle, the square of any side is equal to the sum of the squares of the other two sides, minus twice their product into the cosine of their included angle.*

$$\text{I. To prove} \quad a^2 = b^2 + c^2 - 2bc \cos A. \quad (47)$$

CASE I. *When the included angle A is acute.*



There will be two cases, according as angle B is acute (Fig. 1), or obtuse (Fig. 2).

In each case, draw line CD perpendicular to AB .

In Fig. 1, $BD = c - AD$, and in Fig. 2, $BD = AD - c$.

Squaring, we have in either case,

$$\overline{BD}^2 = \overline{AD}^2 + c^2 - 2c \times AD.$$

Adding \overline{CD}^2 to both members,

$$\overline{BD}^2 + \overline{CD}^2 = \overline{AD}^2 + \overline{CD}^2 + c^2 - 2c \times AD.$$

But, $\overline{BD}^2 + \overline{CD}^2 = a^2$, and $\overline{AD}^2 + \overline{CD}^2 = b^2$.

Also, by § 5, $AD = b \cos A$.

Whence, $a^2 = b^2 + c^2 - 2bc \cos A$.

CASE II. When the included angle A is obtuse.

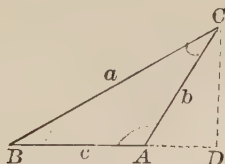


FIG. 8.

Draw line CD perpendicular to AB .

We have $BD = AD + c$.

Squaring, and adding \overline{CD}^2 to both members,

$$\overline{BD}^2 + \overline{CD}^2 = \overline{AD}^2 + \overline{CD}^2 + c^2 + 2c \times AD.$$

But, $\overline{BD}^2 + \overline{CD}^2 = a^2$, and $\overline{AD}^2 + \overline{CD}^2 = b^2$.

And by § 5,

$$AD = b \cos CAD = b \cos (180^\circ - A) = -b \cos A \text{ (§ 32).}$$

Whence, $a^2 = b^2 + c^2 - 2bc \cos A$.

In like manner, $b^2 = c^2 + a^2 - 2ca \cos B$, (48)

and $c^2 = a^2 + b^2 - 2ab \cos C$. (49)

102. To express the cosines of the angles of a triangle in terms of the sides of the triangle.

By (47), $a^2 = b^2 + c^2 - 2bc \cos A$.

Transposing, $2bc \cos A = b^2 + c^2 - a^2$.

Whence,
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}. \quad (50)$$

In like manner,
$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}, \quad (51)$$

and
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}. \quad (52)$$

103. *To express the sines, cosines, and tangents of the half angles of a triangle in terms of the sides of the triangle.*

By (50),

$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - b^2 + 2bc - c^2}{2bc}.$$

Whence by (28),

$$2 \sin^2 \frac{1}{2} A = \frac{a^2 - (b - c)^2}{2bc}.$$

Or,
$$\sin^2 \frac{1}{2} A = \frac{(a - b + c)(a + b - c)}{4bc}.$$

Denoting the sum of the sides, $a + b + c$, by $2s$, we have

$$a - b + c = (a + b + c) - 2b = 2s - 2b = 2(s - b),$$

and $a + b - c = (a + b + c) - 2c = 2s - 2c = 2(s - c).$

Whence,
$$\sin^2 \frac{1}{2} A = \frac{4(s - b)(s - c)}{4bc}.$$

Or,
$$\sin \frac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{bc}}. \quad (53)$$

In like manner,
$$\sin \frac{1}{2} B = \sqrt{\frac{(s - c)(s - a)}{ca}}, \quad (54)$$

and
$$\sin \frac{1}{2} C = \sqrt{\frac{(s - a)(s - b)}{ab}}. \quad (55)$$

Again, by (50),

$$1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + 2bc + c^2 - a^2}{2bc}.$$

Whence by (29),

$$2 \cos^2 \frac{1}{2} A = \frac{(b+c)^2 - a^2}{2bc}.$$

$$\text{Or,} \quad \cos^2 \frac{1}{2} A = \frac{(b+c+a)(b+c-a)}{4bc}.$$

$$\text{But,} \quad b+c+a = 2s,$$

$$\text{and} \quad b+c-a = (b+c+a) - 2a = 2(s-a).$$

$$\text{Whence,} \quad \cos^2 \frac{1}{2} A = \frac{4s(s-a)}{4bc}.$$

$$\text{Or,} \quad \cos \frac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}. \quad (56)$$

$$\text{In like manner,} \quad \cos \frac{1}{2} B = \sqrt{\frac{s(s-b)}{ca}}. \quad (57)$$

$$\text{and} \quad \cos \frac{1}{2} C = \sqrt{\frac{s(s-c)}{ab}}. \quad (58)$$

Dividing (53) by (56), we have,

$$\frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} = \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{bc}{s(s-a)}}.$$

$$\text{Whence by (4),} \quad \tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \quad (59)$$

$$\text{In like manner,} \quad \tan \frac{1}{2} B = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}, \quad (60)$$

$$\text{and} \quad \tan \frac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}. \quad (61)$$

Note. Since each angle of a triangle is less than 180° , its half is less than 90° ; hence, the *positive sign* must be taken before the radical in each formula of § 103.

FORMULÆ FOR THE AREA OF AN OBLIQUE TRIANGLE.

104. CASE I. *Given two sides and their included angle.*

I. When the given parts are b , c , and A .

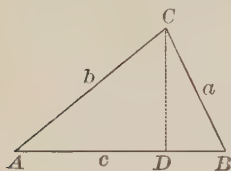


FIG. 1.

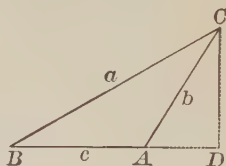


FIG. 2.

There will be two cases, according as A is acute (Fig. 1) or obtuse (Fig. 2).

In each case, draw line CD perpendicular to AB .

Then denoting the area by K , we have by Geometry,

$$2K = c \times CD.$$

But in Fig. 1, $CD = b \sin A$ (§ 5).

And in Fig. 2, $CD = b \sin CAD$

$$= b \sin (180^\circ - A) = b \sin A \text{ (§ 32).}$$

Then in either case,

$$2K = bc \sin A. \quad (62)$$

In like manner, $2K = ca \sin B, \quad (63)$

and $2K = ab \sin C. \quad (64)$

CASE II. *Given a side and all the angles.*

I. When the given parts are a , A , B , and C .

By (64), $2K = ab \sin C.$

But by (41), $\frac{b}{a} = \frac{\sin B}{\sin A}$, or $b = \frac{a \sin B}{\sin A}$.

Whence, $2K = a \times \frac{a \sin B}{\sin A} \times \sin C$

$$= \frac{a^2 \sin B \sin C}{\sin A}. \quad (65)$$

In like manner, $2K = \frac{b^2 \sin C \sin A}{\sin B},$ (66)

and $2K = \frac{c^2 \sin A \sin B}{\sin C}.$ (67)

CASE III. *Given the three sides.*

By (62), $2K = bc \sin A = 2bc \sin \frac{1}{2}A \cos \frac{1}{2}A$, by (22).

Dividing by 2, and substituting the values of $\sin \frac{1}{2}A$ and $\cos \frac{1}{2}A$ from (53) and (56), we have,

$$K = bc \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}. \quad (68)$$

VIII. SOLUTION OF OBLIQUE TRIANGLES.

In the solution of oblique triangles, we may distinguish four cases.

105. CASE I. *Given a side and any two angles.*

The third angle may be found by Geometry, and then by aid of § 99 the remaining sides may be calculated.

The triangle is possible for any values of the given elements, provided the sum of the given angles is $< 180^\circ$.

1. Given $b = 20$, $A = 104^\circ$, $B = 19^\circ$; find C , a , and c

We have $C = 180^\circ - (A + B) = 180^\circ - 123^\circ = 57^\circ$.

$$\text{By § 99,} \quad \frac{a}{b} = \frac{\sin A}{\sin B}, \text{ and } \frac{c}{b} = \frac{\sin C}{\sin B}.$$

$$\text{Then,} \quad a = b \sin A \csc B, \text{ and } c = b \sin C \csc B.$$

$$\text{Whence,} \quad \log a = \log b + \log \sin A + \log \csc B,$$

$$\text{and} \quad \log c = \log b + \log \sin C + \log \csc B.$$

$\log b = 1.3010$ $\log \sin A = 9.9869 - 10$ $\log \csc B = 0.4874$ <hr style="width: 100%;"/> $\log a = 1.7753$ $a = 59.61.$	$\log b = 1.3010$ $\log \sin C = 9.9236 - 10$ $\log \csc B = 0.4874$ <hr style="width: 100%;"/> $\log c = 1.7120$ $c = 51.52.$
--	--

Note. To find the log cosecant of an angle, subtract the log sine from $10 - 10$. To find $\log \sin 104^\circ$, take either $\log \cos 14^\circ$ or $\log \sin 76^\circ$. (See page 7 of the author's New Four Place Logarithmic Tables.)

EXAMPLES.

Solve the following triangles:

2. Given $a = 180$, $A = 38^\circ$, $B = 75^\circ 20'$.

3. Given $b = 8.19$, $B = 52^\circ$, $C = 109^\circ$.

4. Given $c = .0246$, $A = 83^\circ 30'$, $B = 38^\circ 50'$.

5. Given $b = 67.13$, $A = 26^\circ 18'$, $C = 44^\circ 35'$.

6. Given $c = .45924$, $A = 74^\circ 43'$, $C = 61^\circ 29'$.

7. Given $a = 3024$, $B = 133^\circ 34'$, $C = 22^\circ 57'$.

(For additional examples under Case I, see § 112.)

106. CASE II. *Given two sides and their included angle.*

Since one angle is known, the sum of the remaining angles may be found, and then their difference may be calculated by aid of § 100.

Knowing the sum and difference of the angles, the angles themselves may be found, and then the remaining side may be computed as in Case I.

The triangle is possible for any values of the data.

1. Given $a = 82$, $c = 167$, $B = 98^\circ$; find A , C , and b .

By Geometry, $C + A = 180^\circ - B = 82^\circ$.

By § 100,
$$\frac{c + a}{c - a} = \frac{\tan \frac{1}{2}(C + A)}{\tan \frac{1}{2}(C - A)}.$$

Or,
$$\tan \frac{1}{2}(C - A) = \frac{c - a}{c + a} \tan \frac{1}{2}(C + A).$$

Then,

$\log \tan \frac{1}{2}(C - A) = \log(c - a) + \text{colog}(c + a) + \log \tan \frac{1}{2}(C + A).$

$$c - a = 85. \quad \log = 1.9294$$

$$c + a = 249. \quad \text{colog} = 7.6038 - 10$$

$$\frac{1}{2}(C + A) = 41^\circ. \quad \log \tan = 9.9392 - 10$$

$$\log \tan \frac{1}{2}(C - A) = 9.4724 - 10$$

$$\frac{1}{2}(C - A) = 16^\circ 31.7'.$$

Then, $C = \frac{1}{2}(C + A) + \frac{1}{2}(C - A) = 57^\circ 31.7'$,

and $A = \frac{1}{2}(C + A) - \frac{1}{2}(C - A) = 24^\circ 28.3'.$

To find the remaining side, we have by § 99,

$$b = \frac{a \sin B}{\sin A} = a \sin B \csc A.$$

Whence, $\log b = \log a + \log \sin B + \log \csc A.$

$$\log a = 1.9138$$

$$\log \sin B = 9.9958 - 10$$

$$\log \csc A = 0.3828$$

$$\log b = 2.2924$$

$$b = 196.1.$$

EXAMPLES.

Solve the following triangles:

2. Given $a = 67$, $c = 33$, $B = 36^\circ$.

3. Given $a = 986$, $b = 544$, $C = 134^\circ$.

4. Given $b = .149$, $c = .427$, $A = 71^\circ$.

5. Given $a = 3.95$, $b = 6.64$, $C = 68^\circ 30'$.

6. Given $a = 2937$, $c = 6185$, $B = 55^\circ 46'$.

7. Given $b = .01292$, $c = .00286$, $A = 26^\circ 32'$.

(For additional examples under Case II., see § 112.)

107. CASE III. *Given the three sides.*

The angles might be calculated by the formulæ of § 102; but as these are not adapted to logarithmic computation, it is usually more convenient to use the formulæ of § 103.

Each angle should be computed trigonometrically; for we then have a check on the work, since their sum should be 180° .

If all the angles are to be computed, the *tangent* formulæ are the most convenient, since only four different numbers occur in the second members.

If but one angle is required, the *cosine* formula involves the least work.

The triangle is possible for any values of the data, provided no side is greater than the sum of the other two.

If all the angles are required, and the tangent formulæ are used, it is convenient to modify them as follows; by (59),

$$\tan \frac{1}{2} A = \sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}} = \frac{1}{s-a} \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Denoting $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$ by r , we have

$$\tan \frac{1}{2} A = \frac{r}{s-a}.$$

In like manner, $\tan \frac{1}{2} B = \frac{r}{s-b}$, and $\tan \frac{1}{2} C = \frac{r}{s-c}$.

1. Given $a = 2.5$, $b = 2.8$, $c = 2.2$; find A , B , and C .

Here, $2s = a + b + c = 7.5$, and $s = 3.75$.

Then, $s - a = 1.25$, $s - b = .95$, $s - c = 1.55$.

By logarithms,

$$\log r = \frac{1}{2} [\log (s-a) + \log (s-b) + \log (s-c) + \text{colog } s].$$

$$\text{Also,} \quad \log \tan \frac{1}{2} A = \log r - \log (s-a),$$

$$\log \tan \frac{1}{2} B = \log r - \log (s-b),$$

$$\log \tan \frac{1}{2} C = \log r - \log (s-c).$$

$$\log (s-a) = 0.0969$$

$$\log r = 9.8455 - 10$$

$$\log (s-b) = 9.9777 - 10$$

$$\log (s-b) = 9.9777 - 10$$

$$\log (s-c) = 0.1903$$

$$\log \tan \frac{1}{2} B = 9.8678 - 10$$

$$\text{colog } s = 9.4260 - 10$$

$$\frac{1}{2} B = 36^\circ 24.6'.$$

$$2) 19.6909 - 20$$

$$B = 72^\circ 49.2'.$$

$$\log r = 9.8455 - 10$$

$$\log r = 9.8455 - 10$$

$$\log (s-a) = 0.0969$$

$$\log (s-c) = 0.1903$$

$$\log \tan \frac{1}{2} A = 9.7486 - 10$$

$$\log \tan \frac{1}{2} C = 9.6552 - 10$$

$$\frac{1}{2} A = 29^\circ 16.3'.$$

$$\frac{1}{2} C = 24^\circ 19.7'.$$

$$A = 58^\circ 32.6'.$$

$$C = 48^\circ 39.4'.$$

Check, $A + B + C = 180^\circ 1.2'$.

2. Given $a = 7$, $b = 11$, $c = 9.6$; find B .

By § 103,
$$\cos \frac{1}{2} B = \sqrt{\frac{s(s-b)}{ca}}.$$

Or, $\log \cos \frac{1}{2} B = \frac{1}{2}[\log s + \log(s-b) + \text{colog } c + \text{colog } a].$

Here, $2s = 27.6$; whence, $s = 13.8$, $s - b = 2.8$.

$$\log s = 1.1399$$

$$\log(s-b) = 0.4472$$

$$\text{colog } c = 9.0177 - 10$$

$$\text{colog } a = 9.1549 - 10$$

$$2 \overline{)19.7597 - 20}$$

$$\log \cos \frac{1}{2} B = 9.8799 - 10$$

$$\frac{1}{2} B = 40^\circ 40.9', \text{ and } B = 81^\circ 21.8'.$$

EXAMPLES.

Solve the following triangles:

3. Given $a = 5$, $b = 7$, $c = 6$.

4. Given $a = 10$, $b = 9$, $c = 8$.

5. Given $a = .56$, $b = .43$, $c = .89$.

6. Given $a = 70.5$, $b = 56.2$, $c = 63.9$; find A .

7. Given $a = .0292$, $b = .0185$, $c = .0357$; find B .

8. Given $a = 302$, $b = 427$, $c = 674$; find C .

(For additional examples under Case III., see § 112.)

108. CASE IV. *Given two sides, and the angle opposite to one of them.*

It was stated in § 89 that a triangle is in general completely determined when three of its elements are known, provided one of them is a side. The only exceptions occur in Case IV.

To illustrate, let us consider the following example:

Given $a = 52.1$, $b = 61.2$, $A = 31^\circ 26'$, find B , C , and c .

By § 99,
$$\frac{\sin B}{\sin A} = \frac{b}{a}, \text{ or } \sin B = \frac{b \sin A}{a}.$$

Whence,
$$\log \sin B = \log b + \text{colog } a + \log \sin A.$$

$$\log b = 1.7868$$

$$\text{colog } a = 8.2832 - 10$$

$$\log \sin A = 9.7173 - 10$$

$$\log \sin B = 9.7873 - 10$$

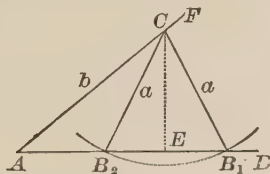
$$B = 37^\circ 47.5', \text{ from the table.}$$

But in finding the angle corresponding, attention must be paid to the fact that an angle and its supplement have the same sine (§ 32).

Therefore another value of B will be $180^\circ - 37^\circ 47.5'$, or $142^\circ 12.5'$; and calling these values B_1 and B_2 , we have

$$B_1 = 37^\circ 47.5', \text{ and } B_2 = 142^\circ 12.5'.$$

The reason for the ambiguity is at once apparent when we attempt to construct the triangle from the data.



We first lay off angle $DAF = 31^\circ 26'$, and on AF take $AC = 61.2$. With C as a centre, and a radius equal to 52.1, describe an arc cutting AD at B_1 and B_2 . Then either of the triangles AB_1C or AB_2C satisfies the given conditions.

The two values of B which were obtained are the values of angles AB_1C and AB_2C , respectively; and it is evident geometrically that these angles are supplementary.

To complete the solution, denote angles ACB_1 and ACB_2 by C_1 and C_2 , and sides AB_1 and AB_2 by c_1 and c_2 , respectively.

Then, $C_1 = 180^\circ - (A + B_1) = 180^\circ - 69^\circ 13.5' = 110^\circ 46.5'$,
and $C_2 = 180^\circ - (A + B_2) = 180^\circ - 173^\circ 38.5' = 6^\circ 21.5'.$

Again, by § 99, $\frac{c_1}{a} = \frac{\sin C_1}{\sin A}$, and $\frac{c_2}{a} = \frac{\sin C_2}{\sin A}$.

Whence, $c_1 = a \sin C_1 \csc A$, and $c_2 = a \sin C_2 \csc A$.

$\log a = 1.7168$	$\log a = 1.7168$
$\log \sin C_1 = 9.9708 - 10$	$\log \sin C_2 = 9.0443 - 10$
$\log \csc A = 0.2827$	$\log \csc A = 0.2827$
<hr/>	<hr/>
$\log c_1 = 1.9703$	$\log c_2 = 1.0438$
$c_1 = 93.40.$	$c_2 = 11.06.$

109. Whenever an angle of an oblique triangle is determined from its *sine*, both the acute and obtuse values must be retained, unless one or both can be shown to be inadmissible; hence there may sometimes be two solutions, sometimes one, and sometimes none, in an example under Case IV.

1. Let the data be a , b , and A , and suppose $b < a$.

By Geometry, B must be $< A$; hence, only the *acute* value of B can be taken; in this case there is but *one* solution.

2. Let the data be a , b , and A , and suppose $b > a$.

Since B must be $> A$, the triangle is impossible unless A is acute.

Again, since $\frac{\sin B}{\sin A} = \frac{b}{a}$, and b is $> a$, $\sin B$ is $> \sin A$.

Hence, both the acute and obtuse values of B are $> A$, and there are *two* solutions, except in the following cases:

If $\log \sin B = 0$, then $\sin B = 1$ (§ 68), and $B = 90^\circ$, and the triangle is a *right* triangle; if $\log \sin B$ is *positive*, then $\sin B$ is > 1 , and the triangle is impossible.

The above results may be stated as follows:

If, of the given sides, that adjacent to the given angle is the *less*, there is but *one* solution, which corresponds to the *acute* value of the opposite angle.

If the side adjacent to the given angle is the *greater*, there are *two* solutions, unless the log sine of the opposite angle is 0 or positive; in which cases there are *one* solution (a *right* triangle), and *no* solution, respectively.

110. We will illustrate these points by examples:

1. Given $a = 7.42$, $b = 3.39$, $A = 105^\circ 13'$; find B .

Since b is $< a$, there is but *one* solution, corresponding to the *acute* value of B .

$$\text{By § 99,} \quad \sin B = \frac{b \sin A}{a}$$

$$\log b = 0.5302$$

$$\text{colog } a = 9.1296 - 10$$

$$\log \sin A = \underline{9.9845 - 10}$$

$$\log \sin B = 9.6443 - 10$$

$$B = 26^\circ 9.6'.$$

2. Given $b = 3$, $c = 2$, $C = 100^\circ$; find B .

Since b is $> c$, and C is obtuse, the triangle is impossible.

3. Given $a = 22.764$, $c = 50$, $A = 27^\circ 4.8'$; find C .

$$\text{We have,} \quad \sin C = \frac{c \sin A}{a}$$

$$\log c = 1.6990$$

$$\text{colog } a = 8.6428 - 10$$

$$\log \sin A = \underline{9.6582 - 10}$$

$$\log \sin C = 0.0000$$

Therefore, $\sin C = 1$, and $C = 90^\circ$.

Here there is but one solution; a *right* triangle.

4. Given $a = .83$, $b = .715$, $B = 61^\circ 47'$; find A

$$\text{We have,} \quad \sin A = \frac{a \sin B}{b}$$

$$\log a = 9.9191 - 10$$

$$\text{colog } b = 0.1457$$

$$\log \sin B = \underline{9.9451 - 10}$$

$$\log \sin A = 0.0099$$

Since $\log \sin A$ is positive, the triangle is impossible.

EXAMPLES.

111. Solve the following triangles:

1. Given $a = 7.3$, $b = 6.6$, $A = 56^\circ$.
2. Given $b = 86$, $c = 159$, $C = 115^\circ$.
3. Given $b = 60.93$, $c = 76.09$, $B = 133^\circ 41'$
4. Given $b = 38$, $c = 48$, $B = 34^\circ$.
5. Given $a = .279$, $c = .227$, $C = 65^\circ 45'$.
6. Given $a = 3215$, $c = 6754$, $A = 28^\circ 26'$.
7. Given $a = .06358$, $c = .08604$, $C = 19^\circ 14'$.
8. Given $a = 186.7$, $b = 394.2$, $B = 114^\circ 28'$.
9. Given $a = .462$, $c = .647$, $A = 31^\circ 7'$.

(For additional examples under Case IV., see § 112.)

MISCELLANEOUS EXAMPLES.

112. Solve the following triangles:

1. Given $a = 934$, $b = 756$, $C = 73^\circ 16'.$
2. Given $c = 8.706$, $B = 38^\circ 45'$, $C = 31^\circ 59'$.
3. Given $a = 61$, $b = 85$, $c = 48$.
4. Given $a = .425$, $c = .454$, $C = 37^\circ 9'$.
5. Given $b = .0479$, $c = .0144$, $A = 121^\circ 28'$.
6. Given $a = 7824$, $c = 3202$, $A = 140^\circ 53'$.
7. Given $b = .0005639$, $A = 44^\circ 24'$, $B = 116^\circ 9'$.
8. Given $a = 1.5$, $b = 1.3$, $c = 1.9$.
9. Given $a = 576$, $b = 813$, $A = 23^\circ 25'.$ ✓
10. Given $b = 2615$, $c = 6086$, $A = 115^\circ 10'$.
11. Given $b = 9.874$, $c = 7.486$, $B = 81^\circ 47'$.
12. Given $a = 71387$, $B = 42^\circ 56'$, $C = 76^\circ 7'$.

13. Given $b = 51.434$, $c = 47.955$. $C = 72^\circ 54'$.
14. Given $a = .008727$, $c = .007065$, $B = 84^\circ 56'$.
15. Given $a = .031$, $b = .024$, $c = .028$.
16. Given $a = .19597$, $b = .13927$, $B = 45^\circ 17'$.
17. Given $a = 3.5374$, $b = 9.6036$, $A = 97^\circ 46'$.
18. Given $a = .40932$, $A = 53^\circ 13'$, $C = 67^\circ 32'$.
19. Given $a = 31.06$, $b = 51.49$, $C = 47^\circ 43'$.
20. Given $a = .019186$, $b = .033728$, $B = 125^\circ 33'$.
21. Given $a = 353.85$, $c = 579.42$, $B = 19^\circ 37'$.
22. Given $b = 24883$, $c = 20609$, $C = 48^\circ 6'$.

AREA OF AN OBLIQUE TRIANGLE.

- 113. 1.** Given $a = 18.063$, $A = 96^\circ 30'$, $B = 35^\circ$; find K

By § 104, $2K = \frac{a^2 \sin B \sin C}{\sin A} = a^2 \sin B \sin C \csc A$.

Whence,

$$\log (2K) = 2 \log a + \log \sin B + \log \sin C + \log \csc A.$$

Here, $C = 180^\circ - (A + B) = 48^\circ 30'$.

$\log a = 1.2568$; multiply by 2 = 2.5136

$$\log \sin B = 9.7586 - 10$$

$$\log \sin C = 9.8745 - 10$$

$$\log \csc A = 0.0028$$

$$\log (2K) = 2.1495$$

$$2K = 141.1, \text{ and } K = 70.55.$$

EXAMPLES.

Find the areas of the following triangles:

2. Given $a = 26.4$, $c = 47.9$, $B = 67^\circ$.
3. Given $a = 8.05$, $B = 65^\circ 30'$, $C = 81^\circ 40'$.
4. Given $a = 7$, $b = 9$, $c = 6$.

5. Given $c = .518$, $A = 67^\circ 45'$, $B = 37^\circ 19'$.
6. Given $b = 15.32$, $c = 36.78$, $A = 105^\circ 43'$.
7. Given $b = 210.6$, $B = 32^\circ 21'$, $C = 108^\circ 56'$.
8. Given $c = .004096$, $A = 17^\circ 45'$, $C = 46^\circ 8'$.
9. Given $a = .73$, $b = .55$, $c = .63$.
10. Given $a = .0006854$, $b = .0009743$, $C = 61^\circ 44'$.
11. Given $a = 7.219$, $A = 23^\circ 33'$, $B = 124^\circ 12'$.
12. Given $a = 5.321$, $c = 8.467$, $B = 152^\circ 51'$.
13. Given $a = 39.5$, $b = 47.3$, $c = 50.8$.
14. Given $b = 250.8$, $A = 77^\circ 53'$, $C = 55^\circ 29'$.
15. Given $b = .19146$, $c = .42829$, $A = 59^\circ 7'$.
16. Given $a = .078$, $b = .091$, $c = .084$.
17. Given $b = 109.41$, $A = 77^\circ 46'$, $B = 43^\circ 32'$.
18. Given $a = 5.7434$, $b = 8.6326$, $C = 129^\circ 17'$.
19. Given $a = 307.4$, $b = 351.9$, $c = 335.7$.
20. Given $a = .0083214$, $A = 34^\circ 44'$, $C = 105^\circ 23'$.
21. Given $a = .064325$, $c = .033777$, $B = 141^\circ 38'$.

MISCELLANEOUS PROBLEMS.

114. 1. To find the distance of an inaccessible object A from a position B , I measure a base-line BC 675 feet in length, and observe the angles ABC and ACB to be $101^\circ 17'$ and $36^\circ 55'$, respectively. Find the distance AB .

2. In a field $ABCD$, the sides AB , BC , CD , and DA are 16, 23, 18, and 29 rods, respectively, and the diagonal AC is 34 rods. Find the area of the field.

3. From the top of a cliff the angles of depression of two stakes in the plain below, in line with the observer, and 725 feet apart, are found to be $35^\circ 10'$ and $19^\circ 40'$, respectively. Find the height of the cliff above the plain.

4. The area of a triangle is 437, and two of its sides are 36 and 43. Find the angle between them.

5. From a point in the same horizontal plane with the base of a tower, the angle of elevation of its top is 42° , and from a point 200 feet farther away, it is 26° . Find the height of the tower, and the distance of its base from each point of observation.

6. Two vessels start at the same point, at the rates of 9.7 and 5.5 miles an hour, respectively, the first due east, and the second due southwest. Find the distance between them at the end of an hour and a half, and the bearing of each from the other.

7. Two sides of a triangle are .85 and .74, and the difference between their opposite angles is $18^\circ 27'$. Solve the triangle.

8. The area of a triangle ABC is 980, its angle A is $56^\circ 20'$, and its side b is 44. Find B , c , and a .

9. The sides of a triangle are 5, 7, and 9, respectively. Find the radius of the inscribed circle.

(By Geometry, the area of a triangle is equal to one-half its perimeter multiplied by the radius of the inscribed circle.)

10. Two sides of a parallelogram are 8 and 5, and include an angle of 61° . Find the diagonals.

11. The diagonals of a field $ABCD$ intersect at E at an angle of 78° . If AE , BE , CE , and DE are 27, 31, 59, and 64 feet, respectively, find the area of the field.

12. The bases of a trapezoid are 49 and 95, and the angles at the extremities of the latter are 64° and 71° . Find the non-parallel sides.

13. Two vessels, A and B , are sailing due northeast. At a certain time, B lies 8 miles due south of A , and at the expiration of an hour 75° east of south. If the rate of A is 6 miles an hour, find the rate of B .

14. From a point in the same horizontal plane with the base of a tower, the angle of elevation of its top is 39° ; and from a point 150 feet vertically above the first, the angle of depression of the top is 43° . Find the height of the tower, and its distance from the first point of observation.

15. From two points on either side of, and in line with, a tower, 300 feet apart, the angles of elevation of its top are observed to be 31° and 27° , respectively. Find the height of the tower.

16. From a point in the same horizontal plane with the base of a tower, the angle of elevation of its top is 22° , and its bearing 31° west of north. From another point 400 feet west of the first, the bearing is 26° east of north. Find the height of the tower.

17. One of the non-parallel sides of a trapezoid is 15, the angle between it and the longer base is 78° , the angle at the other extremity of the longer base is 62° , and the shorter base is 9. Find the other two sides.

18. Two sides of a parallelogram are 103 and 54, and one of the diagonals is 137. Find the angles of the parallelogram, and the other diagonal.

19. From a ship, two lighthouses bear due northwest. After sailing 18 miles in a direction 35° west of south, the lighthouses bear 6° west of north and 9° east of north, respectively. Find the distance between the lighthouses.

20. The sides AB and BC , of quadrilateral $ABCD$, are 9 and 5, respectively, and the angles, A , B , and C are 84° , 109° , and 96° , respectively. Find the sides AD and CD .

21. From a position at the foot of a hill surmounted by a tower, the angle of elevation of the top of the tower is 31° . After walking 1260 feet toward the foot of the tower, up a slope whose angle with a horizontal plane is 29° , the tower subtends an angle of 25° . How far is the top of the tower above the horizontal plane of the foot of the hill?

22. The sides AB , BC , and CD , of quadrilateral $ABCD$, are 23, 41, and 36, respectively, and the angles B and C are 116° and 131° , respectively. Find the side AD and the angles A and D .

23. The diagonals of a parallelogram are 58 and 92, respectively, and intersect at an angle of 55° . Find the sides and angles of the parallelogram.

24. From two points on the slope of a hill, in the same vertical plane with the summit, the angles of elevation of the top are 11° and 18° , respectively. The points are 300 feet apart, and the second 40 feet above the horizontal plane of the first. How far is the top of the hill above the horizontal plane of the first point?

25. To find the distance between two inaccessible buoys, A and B , a line CD , 150 feet in length, is measured on the shore. At C the angles ACD and BCD are observed to be 83° and 69° , respectively, and at D the angles ADC and BDC are observed to be 74° and 97° , respectively. Find the distance AB .

26. A bluff, with a lighthouse on its edge, is observed from a boat, the angle of elevation of the top of the lighthouse being 25° . After rowing 1000 feet directly toward the lighthouse, the angles of elevation of its top and bottom are found to be 53° and 39° , respectively. Find the height of the bluff, and of the lighthouse.

APPENDIX TO PLANE TRIGONOMETRY.

PARALLEL SAILING.

In **Parallel Sailing**, a vessel sails along a *parallel of latitude* from one position to another in the same latitude.

Let O' be the centre of the earth ; P , the north pole ; PA' and PB' , meridians ; $A'B'$, the equator ; AB , a parallel, intersecting PA' at A and PB' at B , and having its centre at O ; draw lines OA , OB , $O'A'$, $O'B'$, $O'A$, and $O'P$.

The number of *geographical miles** in AB is called the *departure* between A and B .

By geometry,

$$\frac{\text{arc } A'B'}{\text{arc } AB} = \frac{O'A'}{OA} = \frac{O'A}{OA} = \sec O'AO.$$

Whence, $\text{arc } A'B' = \text{arc } AB \times \sec A'O'A$.

That is, difference of longitude between A and B = departure between A and B \times sec lat. A .

Ex. A vessel whose position is lat. $25^{\circ} 20'$ N., lon. $36^{\circ} 10'$ W., sails due west 140 geographical miles. Find the longitude of the position reached.

Here, $\text{diff. long.} = 140 \sec 25^{\circ} 20'.$

$$\log 140 = 2.1461$$

$$\log \sec 25^{\circ} 20' = 0.0439$$

$$\underline{2.1900} = \log 154.9.$$

Then, $\text{longitude} = 36^{\circ} 10' + 154.9' = 38^{\circ} 44.9' \text{ W.}$

If a vessel sails from a position whose latitude and longitude are known, due west or due east, until it is in a certain longitude, and the distance in geographical miles is required, it follows from the above that

$$\text{departure} = \text{diff. long.} \div \cos \text{latitude.}$$

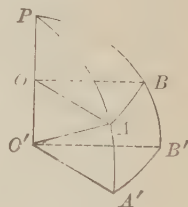


FIG. 1.

* A *geographical mile* is a minute of arc on the equator.

† In geographical miles. In all cases hereafter in which the word "mile" is used, it will be understood to mean a geographical mile.

EXAMPLES.

1. A vessel in lat. $36^{\circ} 48'$ N., lon. $56^{\circ} 15'$ W., sails due east 226 miles. Find the longitude of the position reached.

2. A vessel in lat. $48^{\circ} 54'$ N., lon. $10^{\circ} 55'$ W., sails due west until it is in lon. $15^{\circ} 12'$ W. Find the number of miles sailed.

MIDDLE LATITUDE SAILING.

If a vessel sails from A to B in such a manner that its path makes the *same angle* with every meridian that it crosses, it traverses a curve called a *rhumb line*.

Let BC be a parallel drawn through B , meeting the meridian PA at C ; then, CA is the difference in latitude between A and B .

If the distance traversed is small, we may take ABC as a plane triangle, right-angled at C , having its angle A equal to the course.

CB is the departure and AB the distance; then,

$$\begin{aligned}\text{dep. } CB &= \text{dist. } AB \times \sin A, \\ \text{diff. lat. } AC &= \text{dist. } AB \times \cos A.\end{aligned}$$

If AB is too long to neglect the curvature of the earth, it may be divided into parts, AD , DE , etc., each being of such length that the curvature of the earth may be neglected in it.

Draw parallels DG , EH , etc., meeting meridians PA , PD , etc., at G , H , etc., respectively.

Then, ADG , DEH , etc., may be regarded as plane triangles, right-angled at G , H , etc., having $\angle DAG = \angle EDH$ etc. = course.

$$\begin{aligned}\text{Then, } GD + HE + \dots &= AD \sin A + DE \sin A + \dots \\ &= (AD + DE + \dots) \sin A.\end{aligned}$$

$$\text{Or, total departure} = \text{total distance} \times \sin A.$$

$$\begin{aligned}\text{Also, } AG + DH + \dots &= AD \cos A + DE \cos A + \dots \\ &= (AD + DE + \dots) \cos A.\end{aligned}$$

$$\text{Or, total diff. lat.} = \text{total distance} \times \cos A.$$

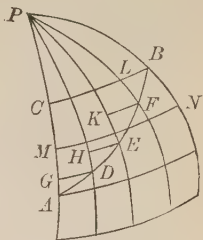


FIG. 2.

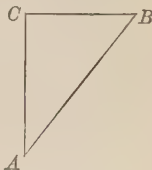


FIG. 3.

The above relations may be represented by triangle ABC , right-angled at C , having its sides AB , BC , and CA equal to the distance, departure, and difference of latitude, respectively, and its $\angle A$ equal to the course.

In **Middle Latitude Sailing**, we take the total departure as measured on the parallel MN (Fig. 2), midway between the parallels of A and B ; it is evident from the figure that MN is approximately equal to the sum of GD , HE , KF , and LB .

(This is sufficiently accurate if the run is not of great length, nor too far away from the equator.)

By the principles of parallel sailing, we have

$$\text{dep. } MN = \text{diff. lon. } AB \times \cos \text{ lat. } M.$$

The latitude of M is one-half the sum of the latitudes of A and B ; this is called the *middle latitude*.

Then, $\text{dep.} = \text{diff. lon.} \times \cos \text{ middle lat.}$

This may be represented graphically by annexing to Fig. 4 the right triangle BCD , having its hypotenuse BD equal to the difference of longitude, and its angle CBD equal to the middle latitude.

Any problem in middle latitude sailing may be solved by constructing a figure, as above, and noting what is given and required; the letter A should be at the starting-point of the vessel.

Ex. A ship, in lat. $42^\circ 30'$ N., lon. $58^\circ 51'$ W., sails S. $33^\circ 45'$ E. 300 miles. Find the latitude and longitude of the position reached.

Here, $A = 33^\circ 45'$, $AB = 300$.

Then, $\text{diff. lat. } AC = 300 \cos 33^\circ 45'$.

$$\log 300 = 2.4771$$

$$\log \cos 33^\circ 45' = 9.9198 - 10$$

$$\frac{2.3969}{} = \log 249.4.$$

Then, $\text{lat.} = 42^\circ 30' - 249.4' = 38^\circ 20.6' \text{ N.}$

$\text{mid. lat.} = \frac{1}{2}(42^\circ 30' + 38^\circ 20.6') = 40^\circ 25.3'.$

$\text{diff. lon. } BD = BC \times \sec \text{ mid. lat.}$

$$= 300 \sin 33^\circ 45' \sec 40^\circ 25.3'.$$

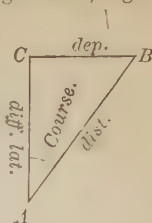


FIG. 4.

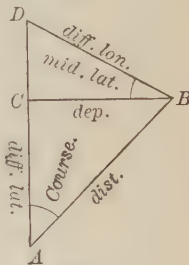


FIG. 5.

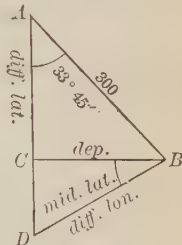


FIG. 6.

* A geographical mile is equal to a minute of latitude.

$$\log 300 = 2.4771$$

$$\log \sin 33^\circ 45' = 9.7448 - 10$$

$$\log \sec 40^\circ 25.3' = 0.1185$$

$$\frac{2.3404}{} = \log 219.$$

Then, $\text{lon.} = 58^\circ 51' - 219' = 55^\circ 12' \text{ W.}$

EXAMPLES.

1. A ship in lat. $26^\circ 15' \text{ N.}$, lon. $61^\circ 43' \text{ W.}$, sails N.W. 253 miles. Find the latitude and longitude of the position reached.

2. A ship sails from a position whose lat. is $49^\circ 56' \text{ N.}$, lon. $15^\circ 16' \text{ W.}$, to another whose lat. is $47^\circ 18' \text{ N.}$, lon. $20^\circ 10' \text{ W.}$ Find the course and distance.

(The difference in latitude and difference in longitude (in miles) are known, and the middle latitude.)

3. A vessel in lat. 37° N. , lon. $32^\circ 16' \text{ W.}$, sails N. $36^\circ 56' \text{ W.}$, and is in lat. 41° N. Find the distance and the longitude of the position reached.

4. A ship in lat. $42^\circ 30' \text{ N.}$, lon. $58^\circ 51' \text{ W.}$, sails, in a direction between south and east, until her departure is 163 miles and her latitude $38^\circ 22' \text{ N.}$ Find her course and distance and the longitude of the position reached.

5. A vessel in lat. $47^\circ 44' \text{ N.}$, lon. $32^\circ 44' \text{ W.}$, sails 171 miles, in a direction between north and east, until her latitude is $50^\circ 2' \text{ N.}$ Find her course and the longitude of the position reached.

6. What is the course and distance in sailing from Pernambuco (lat. $3^\circ 27' \text{ S.}$, lon. $34^\circ 50' \text{ W.}$) to the Cape of Good Hope (lat. $34^\circ 23' \text{ S.}$, lon. $18^\circ 29' \text{ E.}$) ?

7. A ship in lat. $47^\circ 15' \text{ N.}$, lon. $20^\circ 48' \text{ W.}$, sails, in a direction between south and west, 208 miles, until the departure is 162 miles. Find the course and the latitude and longitude of the position reached.

8. A vessel in lat. $51^\circ 16' \text{ S.}$, lon. $34^\circ 13' \text{ E.}$, sails E.N.E. until the departure is 156 miles. Find the distance sailed and the latitude and longitude of the position reached.

TRAVERSE SAILING.

In **Traverse Sailing**, a vessel sails from one position to another on two or more different courses.

The path which it follows is called a *traverse*.

Each portion of the traverse is worked out independently and the results combined.

Ex. A vessel sails from a position in lat. $25^{\circ} 20' N.$, lon. $64^{\circ} 30' W.$, E.N.E. 135 miles, and then S. by E. 148 miles. Find the latitude and longitude of the position reached and its bearing and distance from the starting-point.

Proceeding in the same manner as in the illustrative example in middle latitude sailing, we find that, on the first course, the position reached is in lat. $26^{\circ} 11.66' N.$, lon. $62^{\circ} 11.5' W.$

In the same manner we find that, on the second course, the position reached is in lat. $23^{\circ} 46.56' N.$, lon. $61^{\circ} 39.65' W.$

Then, as in Ex. 3 under middle latitude sailing, we find that, if sailing from lat. $25^{\circ} 20' N.$, lon. $64^{\circ} 30' W.$, to lat. $23^{\circ} 46.56' N.$, lon. $61^{\circ} 39.65' W.$, the course would be S. $58^{\circ} 54.3' E.$, and the distance 180.9 miles.

(The traverse would not be worked out by the principles of middle latitude sailing unless its parts were of such length that the curvature of the earth could not be neglected.)

EXAMPLES.

1. A vessel in lat. $21^{\circ} 45' S.$, lon. $73^{\circ} 10' E.$, sails S.W. by S. 158 miles, then N.N.W. 172 miles. Find the latitude and longitude of the position reached and its bearing and distance from the starting-point.

2. A ship in lat. $39^{\circ} N.$, lon. $42^{\circ} 30' W.$, sails N. $15^{\circ} 23' E.$ 161 miles, then S. $72^{\circ} 14' E.$ 186 miles, then S. $48^{\circ} 42' W.$ 195 miles. Find the latitude and longitude of the position reached and its bearing and distance from the starting-point.

ANSWERS.

Parallel Sailing. — 1. $51^{\circ} 32.7' W.$ 2. 168.9.

Middle Latitude Sailing. — 1. Lat. $29^{\circ} 13.9' N.$; lon. $65^{\circ} 5.1' W.$
 2. Course, S. $50^{\circ} 53.1' W.$; distance, 250.5 miles. 3. Distance, 300.3 miles; lon. $36^{\circ} 8.2' W.$ 4. Course, S. $33^{\circ} 18.9' E.$; distance, 296.7 miles; lon. $55^{\circ} 16.8' W.$ 5. Course, N. $36^{\circ} 11.1' E.$; lon. $30^{\circ} 10.4' W.$
 6. Course, S. $58^{\circ} 28.6' E.$; distance, 3550 miles. 7. Course, S. $51^{\circ} 9' W.$; lat. $45^{\circ} 4.5' N.$; lon. $24^{\circ} 41.9' W.$ 8. Distance, 168.8 miles; lat. $50^{\circ} 11.4' S.$; lon. $38^{\circ} 19.4' E.$

Traverse Sailing. — 1. Lat. $21^{\circ} 17.5' S.$; lon. $70^{\circ} 23.52' E.$; distance, 157.3 m.; course, N. $79^{\circ} 55.8' W.$ 2. Lat. $38^{\circ} 29.84' N.$; lon. $40^{\circ} 48.91' W.$; distance, 84.42 m.; course, S. $69^{\circ} 3.9' E.$

SPHERICAL TRIGONOMETRY.



IX. GEOMETRICAL PRINCIPLES.

115. If a triedral angle be formed with its vertex at the centre of a sphere, it intercepts on the surface a *spherical triangle*.

The triangle is bounded by three arcs of great circles, called its *sides*, which measure the face angles of the triedral angle.

The *angles* of the spherical triangle are the spherical angles formed by the adjacent sides; and each is equal to the angle between two straight lines drawn, one in the plane of each of its sides, perpendicular to the intersection of these planes at the same point.

The sides of a spherical triangle are usually expressed in degrees.

116. A spherical triangle is called *right* when it has a right angle; *quadrantal* when it has one side a quadrant.

117. *Spherical Trigonometry* treats of the trigonometric relations between the sides and angles of a spherical triangle.

The face and diedral angles of the triedral angle are not altered by varying the radius of the sphere; and hence the relations between the sides and angles of a spherical triangle are independent of the length of the radius.

118. We shall limit ourselves in the present work to such triangles as are considered in Geometry, where each angle is less than two right angles, and each side less than the semi-circumference of a great circle; that is, where each element is less than 180° .

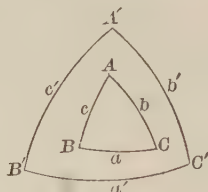
119. The proofs of the following properties of spherical triangles may be found in any treatise on Solid Geometry:

1. Any side of a spherical triangle is less than the sum of the other two sides.

2. The sum of the sides of a spherical triangle is less than 360° .

3. The sum of the angles of a spherical triangle is greater than 180° , and less than 540° .

4. If $A'B'C'$ is the polar triangle of spherical triangle ABC , that is, if A , B , and C are poles of sides $B'C'$, $C'A'$, and $A'B'$, respectively, then ABC is the polar triangle of spherical triangle $A'B'C'$.



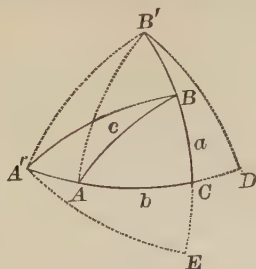
5. In two polar triangles, each angle of one is measured by the supplement of that side of the other of which it is the pole; that is,

$$\begin{aligned} a' &= 180^\circ - A. & b' &= 180^\circ - B. & c' &= 180^\circ - C. \\ A' &= 180^\circ - a. & B' &= 180^\circ - b. & C' &= 180^\circ - c. \end{aligned}$$

6. If two angles of a spherical triangle are unequal, the sides opposite are unequal, and the greater side lies opposite the greater angle; conversely, if two sides of a spherical triangle are unequal, the angles opposite are unequal, and the greater angle lies opposite the greater side.

120. A spherical triangle is called *tri-rectangular* when it has three right angles; each side is a quadrant, and each vertex is the pole of the opposite side.

121. I. Let C be the right angle of right spherical triangle ABC , and suppose $a < 90^\circ$ and $b < 90^\circ$.



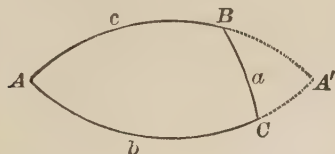
Complete the tri-rectangular triangle $A'B'C$; also, since B' is the pole of AC , and A' of BC , construct the tri-rectangular triangles $AB'D$ and $A'BE$.

Then since B lies within triangle $AB'D$, AB or c is $< 90^\circ$.

Since BC is $< B'C$, $\angle A$ is $< \angle B'AD$, or $< 90^\circ$.

Since AC is $< A'C$, $\angle B$ is $< \angle A'BE$, or $< 90^\circ$.

II. Suppose $a < 90^\circ$ and $b > 90^\circ$.



Complete the lune $ABA'C$.

Then in right triangle $A'BC$, $A'C = 180^\circ - b$.

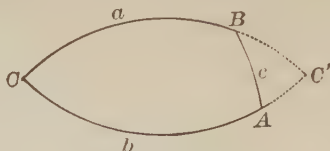
That is, sides a and $A'C$ of triangle $A'BC$ are each $< 90^\circ$; and by I., $A'B$ and angles A' and $A'BC$ are each $< 90^\circ$.

But, $c = 180^\circ - A'B$, $A = A'$, and $B = 180^\circ - A'BC$.

Whence, c is $> 90^\circ$, $A < 90^\circ$, and $B > 90^\circ$.

Similarly, if a is $> 90^\circ$ and $b < 90^\circ$, then c is $> 90^\circ$, $A > 90^\circ$, and $B < 90^\circ$.

III. Suppose $a > 90^\circ$ and $b > 90^\circ$.



Complete the lune $ACBC'$.

Then in right triangle ABC' ,

$$AC' = 180^\circ - b, \text{ and } BC' = 180^\circ - a.$$

That is, sides AC' and BC' of triangle ABC' are each $< 90^\circ$; and by I., AB and angles BAC' and ABC' are each $< 90^\circ$.

But, $A = 180^\circ - BAC'$, and $B = 180^\circ - ABC'$.

Whence, c is $< 90^\circ$, $A > 90^\circ$, and $B > 90^\circ$.

Hence, in any right spherical triangle:

1. *If the sides about the right angle are in the same quadrant, the hypotenuse is $< 90^\circ$; if they are in different quadrants, the hypotenuse is $> 90^\circ$.*

2. *An angle is in the same quadrant as its opposite side.*

122. In the figure of § 119, we have, by § 119, 1, $x' < b' + c'$.

Putting for a' , b' , and c' the values given in § 119, 5, we have

$$180^\circ - A < 180^\circ - B + 180^\circ - C, \text{ or } B + C - A < 180^\circ.$$

Again, by § 118, $B + C + 180^\circ > A$.

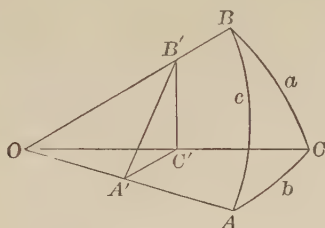
Whence, $B + C - A$ is $> -180^\circ$.

Therefore, $B + C - A$ is between 180° and -180° .

Similarly, $C + A - B$ and $A + B - C$ are between 180° and -180° .

X. RIGHT SPHERICAL TRIANGLES.

123. Let C be the right angle of right spherical triangle ABC , and O the centre of the sphere.



Draw radii OA , OB , and OC .

At any point A' of OA , draw lines $A'B'$ and $A'C'$ in planes OAB and OAC , respectively, perpendicular to OA , meeting OB and OC at B' and C' , respectively; also, draw line $B'C'$.

Then, OA is perpendicular to plane $A'B'C'$.

Whence, each of the planes $A'B'C'$ and OBC is perpendicular to plane OAC , and hence $B'C'$ is perpendicular to OAC .

Therefore, $B'C'$ is perpendicular to $A'C'$ and OC' .

By § 115, sides a , b , and c measure angles BOC , COA , and AOB , respectively, and the angle A of the spherical triangle is equal to angle $B'A'C'$.

In right triangle $OA'B'$, we have

$$\cos c = \cos A'OB' = \frac{OA'}{OB'} = \frac{OC'}{OB'} \times \frac{OA'}{OC'}.$$

But in right triangles $OB'C'$ and $OC'A'$,

$$\frac{OC'}{OB'} = \cos a, \text{ and } \frac{OA'}{OC'} = \cos b.$$

Whence,

$$\cos c = \cos a \cos b.$$

(69)

$$\text{Again, } \sin A = \sin B'A'C' = \frac{B'C'}{A'B'} = \frac{\frac{B'C'}{OB'}}{\frac{A'B'}{OB'}} = \frac{\sin a}{\sin c}. \quad (70)$$

$$\text{And, } \cos A = \cos B'A'C' = \frac{A'C'}{A'B'} = \frac{\frac{A'C'}{OA'}}{\frac{A'B'}{OA'}} = \frac{\tan b}{\tan c}. \quad (71)$$

$$\text{In like manner, } \sin B = \frac{\sin b}{\sin c}, \quad (72)$$

$$\text{and } \cos B = \frac{\tan a}{\tan c}. \quad (73)$$

124. From (70) and (71), we obtain

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin a}{\sin c} \times \frac{\tan c}{\tan b} = \frac{\sin a}{\cos c \tan b}.$$

$$\text{Whence by (69), } \tan A = \frac{\sin a}{\cos a \cos b \tan b} = \frac{\tan a}{\sin b}. \quad (74)$$

$$\text{In like manner, } \tan B = \frac{\tan b}{\sin a}. \quad (75)$$

125. By (4), $\sin a = \cos a \tan a$; then (70) may be written

$$\sin A = \frac{\cos a \tan a}{\cos c \tan c} = \frac{\frac{\tan a}{\cos a}}{\frac{\tan c}{\cos c}}.$$

Whence by (69) and (73),

$$\sin A = \frac{\cos B}{\cos b}. \quad (76)$$

$$\text{In like manner, } \sin B = \frac{\cos A}{\cos a}. \quad (77)$$

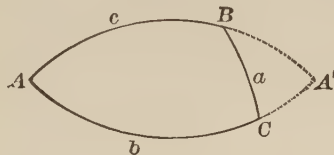
126. From (69), (76), and (77), we have

$$\cos c = \cos a \cos b = \frac{\cos A}{\sin B} \times \frac{\cos B}{\sin A} = \cot A \cot B. \quad (78)$$

127. The proofs of § 123 cannot be regarded as general, for in the construction of the figure we have assumed a and b , and therefore c and A (§ 121), to be less than 90° .

To prove formulæ (69) to (73) universally, we must consider two additional cases:

CASE I. When one of the sides a and b is $< 90^\circ$, and the other $> 90^\circ$.



In right spherical triangle ABC , let a be $< 90^\circ$ and $b > 90^\circ$. Complete the lune $ABA'C$; then, in spherical triangle $A'BC$,

$$A'B = 180^\circ - c, \quad A'C = 180^\circ - b, \quad A' = A, \quad \text{and} \quad A'BC = 180^\circ - B.$$

But by § 121, c is $> 90^\circ$, $A < 90^\circ$, and $B > 90^\circ$.

Hence, each element, except the right angle, of right spherical triangle $A'BC$ is $< 90^\circ$; and we have by § 123,

$$\cos A'B = \cos a \cos A'C,$$

$$\sin A' = \frac{\sin a}{\sin A'B},$$

$$\sin A'BC = \frac{\sin A'C}{\sin A'B},$$

$$\cos A' = \frac{\tan A'C}{\tan A'B},$$

$$\cos A'BC = \frac{\tan a}{\tan A'B}.$$

Putting for $A'B$, $A'C$, A' and $A'BC$ their values, we have

$$\cos (180^\circ - c) = \cos a \cos (180^\circ - b),$$

$$\sin A = \frac{\sin a}{\sin(180^\circ - c)}, \quad \sin(180^\circ - B) = \frac{\sin(180^\circ - b)}{\sin(180^\circ - c)},$$

$$\cos A = \frac{\tan(180^\circ - b)}{\tan(180^\circ - c)}, \quad \cos(180^\circ - B) = \frac{\tan a}{\tan(180^\circ - c)}.$$

Whence, by § 32, $-\cos c = \cos a(-\cos b)$,

$$\sin A = \frac{\sin a}{\sin c}, \quad \sin B = \frac{\sin b}{\sin c},$$

$$\cos A = \frac{-\tan b}{-\tan c}, \quad -\cos B = \frac{\tan a}{-\tan c};$$

and we obtain formulæ (69) to (73) as before.

In like manner, the formulæ may be proved to hold when a is $> 90^\circ$ and $b < 90^\circ$.

CASE II. *When both a and b are $> 90^\circ$.*



In right spherical triangle ABC , let a and b be $> 90^\circ$.
Complete the lune $ACBC'$.

By § 121, c is $< 90^\circ$, $A > 90^\circ$, and $B > 90^\circ$.

Hence, each element, except the right angle, of right spherical triangle ABC' is $< 90^\circ$; and we have by § 123,

$$\cos c = \cos AC' \cos BC',$$

$$\sin BAC = \frac{\sin BC'}{\sin c}, \quad \sin ABC' = \frac{\sin AC'}{\sin c},$$

$$\cos EAC'' = \frac{\tan AC'}{\tan c}, \quad \cos ABC' = \frac{\tan BC'}{\tan c}.$$

Putting for AC' , BC' , BAC' and ABC' their values,

$$\cos c = \cos(180^\circ - a) \cos(180^\circ - b),$$

$$\sin(180^\circ - A) = \frac{\sin(180^\circ - a)}{\sin c},$$

$$\cos(180^\circ - A) = \frac{\tan(180^\circ - b)}{\tan c},$$

$$\sin(180^\circ - B) = \frac{\sin(180^\circ - b)}{\sin c},$$

$$\cos(180^\circ - B) = \frac{\tan(180^\circ - a)}{\tan c}.$$

Whence, by § 32, $\cos c = (-\cos a)(-\cos b)$,

$$\sin A = \frac{\sin a}{\sin c}, \quad \sin B = \frac{\sin b}{\sin c},$$

$$-\cos A = \frac{-\tan b}{\tan c}, \quad -\cos B = \frac{-\tan a}{\tan c};$$

and we obtain formulæ (69) to (73), as before.

128. The formulæ of §§ 123 to 126 are collected below for convenience of reference:

$$\cos c = \cos a \cos b.$$

$$\sin A = \frac{\sin a}{\sin c}, \quad \sin B = \frac{\sin b}{\sin c}.$$

$$\cos A = \frac{\tan b}{\tan c}, \quad \cos B = \frac{\tan a}{\tan c}.$$

$$\tan A = \frac{\tan a}{\sin b}, \quad \tan B = \frac{\tan b}{\sin a}.$$

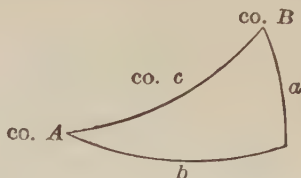
$$\sin A = \frac{\cos B}{\cos b}, \quad \sin B = \frac{\cos A}{\cos a}.$$

$$\cos c = \cot A \cot B.$$

The student should compare the formulæ for the sines, cosines, and tangents of A and B with the corresponding formulæ in §§ 2 and 4.

129. Napier's Rules of Circular Parts.

These are two rules which include all the formulæ of § 128.



In any right spherical triangle, the elements a and b , and the complements of elements A , B , and c (written in abbreviated form, $\text{co. } A$, $\text{co. } B$, and $\text{co. } c$), are called the *circular parts*.

If we suppose them arranged in the order in which the letters occur in the triangle, any one of the five may be taken and called the *middle part*; the two immediately adjacent are called the *adjacent parts*, and the remaining two the *opposite parts*.

Then Napier's rules are:

I. *The sine of the middle part is equal to the product of the tangents of the adjacent parts.*

II. *The sine of the middle part is equal to the product of the cosines of the opposite parts.*

130. Napier's rules may be proved by taking each circular part in succession as the middle part, and showing that the results agree with the formulæ of § 128.

1. If a be taken as the middle part, b and $\text{co. } B$ are the adjacent parts, and $\text{co. } c$ and $\text{co. } A$ the opposite parts.

Then the rules give

$$\sin a = \tan b \tan (\text{co. } B), \text{ and } \sin a = \cos (\text{co. } c) \cos (\text{co. } A).$$

Or by § 31, $\sin a = \tan b \cot B$, and $\sin a = \sin c \sin A$; which are equivalent to (75) and (70).

2. If b be taken as the middle part, a and $\text{co. } A$ are the adjacent parts, and $\text{co. } c$ and $\text{co. } B$ the opposite parts.

Then, $\sin b = \tan a \tan (\text{co. } A) = \tan a \cot A$,
and $\sin b = \cos (\text{co. } c) \cos (\text{co. } B) = \sin c \sin B$;
which are equivalent to (74) and (72).

3. If $\text{co. } c$ be taken as the middle part, $\text{co. } A$ and $\text{co. } B$ are the adjacent parts, and a and b the opposite parts.

Then,
 $\sin (\text{co. } c) = \tan (\text{co. } A) \tan (\text{co. } B)$, and $\sin (\text{co. } c) = \cos a \cos b$.

Or, $\cos c = \cot A \cot B$, and $\cos c = \cos a \cos b$;
which agree with (78) and (69).

4. If $\text{co. } A$ be taken as the middle part, b and $\text{co. } c$ are the adjacent parts, and a and $\text{co. } B$ the opposite parts.

Then,
 $\sin (\text{co. } A) = \tan b \tan (\text{co. } c)$, and $\sin (\text{co. } A) = \cos a \cos (\text{co. } B)$.

Or, $\cos A = \tan b \cot c$, and $\cos A = \cos a \sin B$;
which are equivalent to (71) and (77).

5. If $\text{co. } B$ be taken as the middle part, a and $\text{co. } c$ are the adjacent parts, and b and $\text{co. } A$ the opposite parts.

Then,
 $\sin (\text{co. } B) = \tan a \tan (\text{co. } c)$, and $\sin (\text{co. } B) = \cos b \cos (\text{co. } A)$.

Or, $\cos B = \tan a \cot c$, and $\cos B = \cos b \sin A$;
which are equivalent to (73) and (76).

Writers on Trigonometry differ as to the practical value of Napier's rules; but in the opinion of the highest authorities, it seems to be regarded as preferable to attempt to remember the formulæ by comparing them with the analogous formulæ for plane right triangles, as stated in § 128.

SOLUTION OF RIGHT SPHERICAL TRIANGLES

131. To solve a right spherical triangle, two elements must be given in addition to the right angle.

There may be six cases :

1. *Given the hypotenuse and an adjacent angle.*
2. *Given an angle and its opposite side.*
3. *Given an angle and its adjacent side.*
4. *Given the hypotenuse and another side.*
5. *Given the two sides a and b .*
6. *Given the two angles A and B .*

132. Each of these cases may be solved by the aid of § 12.

If any two elements are given, the formula for computing each remaining element may be found as follows :

Take the formula which involves the given parts and the required part.

If *all* the remaining elements are required, the following rule will be found convenient :

Take the three formulæ which involve the given parts.

133. It is convenient to have a check on the logarithmic work, which may always be done without the necessity of looking out any new logarithms.

Examples of this will be found in § 136.

The check formula for any particular case may be selected from the set in § 128 by the following rule :

Take the formula which involves the three required parts.

Note. If Napier's rules are used, the following rule will indicate which of the circular parts corresponding to the given elements and any required element is to be regarded as the middle part.

If these three circular parts are adjacent, take the middle one as the middle part, and the others are then adjacent parts.

If they are not adjacent, take the part which is not adjacent to either of the others as the middle part, and the others are then opposite parts.

For the check formula, proceed as above with the circular parts corresponding to the three required elements.

Thus, if c and A are the given elements,

1. To find a , consider the circular parts a , $\text{co. } c$, and $\text{co. } A$; of these, a is the middle part, and $\text{co. } c$ and $\text{co. } A$ are opposite parts. Then, by Napier's rules,

$$\sin a = \cos (\text{co. } c) \cos (\text{co. } A) = \sin c \sin A.$$

2. To find b , the circular parts are b , $\text{co. } c$, and $\text{co. } A$; in this case $\text{co. } A$ is the middle part, and b and $\text{co. } c$ are adjacent parts. Then,

$$\sin (\text{co. } A) = \tan b \tan (\text{co. } c), \text{ or } \cos A = \tan b \cot c.$$

3. To find B , the circular parts are $\text{co. } B$, $\text{co. } c$, and $\text{co. } A$; $\text{co. } c$ is the middle part, and $\text{co. } A$ and $\text{co. } B$ are adjacent parts. Then,

$$\sin (\text{co. } c) = \tan (\text{co. } A) \tan (\text{co. } B), \text{ or } \cos c = \cot A \cot B.$$

4. For the check formula, the circular parts are a , b , and $\text{co. } B$; a is the middle part, and b and $\text{co. } B$ are adjacent parts. Then,

$$\sin a = \tan b \tan (\text{co. } B) = \tan b \cot B.$$

134. In solving spherical triangles, careful attention must be paid to the *algebraic signs* of the functions; the cosines, tangents, and cotangents of angles between 90° and 180° being taken *negative* (§ 21).

It is convenient to place the sign of each function just above or below it, as shown in the examples of § 136; the sign of the function in the first member being then determined in accordance with the principle that, in multiplication or division, like signs produce $+$, and unlike signs produce $-$.

Note. In the examples after the first of § 136, the signs are omitted in every case where both functions in the second member are positive.

135. In finding angles corresponding, if the function is a cosine, tangent, or cotangent, its sign determines whether the angle is acute or obtuse; that is, if it is +, the angle is acute; and if it is —, the angle is obtuse, and the *supplement* of the acute angle obtained from the tables must be taken (§ 32).

If the function is a sine, since the sine of an angle is equal to the sine of its supplement (§ 32), both the acute angle obtained from the tables and its supplement must be retained as solutions, unless the ambiguity can be removed by the principles of § 121.

EXAMPLES.

136. 1. Given $B = 33^\circ 50'$, $a = 108^\circ$; find A , b , and c .

By the rule of § 132, the formulæ from § 128 are,

$$\sin B = \frac{\cos A}{\cos a}, \quad \tan B = \frac{\tan b}{\sin a}, \quad \cos B = \frac{\tan a}{\tan c}.$$

$$\text{That is, } \cos A = \cos a \sin B, \quad \tan b = \sin a \tan B, \quad \tan c = \frac{\tan a}{\cos B}.$$

$$\text{Hence,} \quad \log \cos A = \log \cos a + \log \sin B.$$

$$\log \tan b = \log \sin a + \log \tan B.$$

$$\log \tan c = \log \tan a - \log \cos B.$$

Since $\cos A$ and $\tan c$ are negative, the *supplements* of the acute angles obtained from the tables must be taken (§ 135).

Note 1. When the supplement of the angle obtained from the tables is to be taken, it is convenient to write 180° minus the element in the first member, as shown below in the cases of A and c .

By the rule of § 133, the check formula for this case is

$$\cos A = \frac{\tan b}{\tan c}, \quad \text{or} \quad \log \cos A = \log \tan b - \log \tan c.$$

The values of $\log \tan b$ and $\log \tan c$ may be taken from the first part of the work, and their difference should be equal to the result previously found for $\log \cos A$.

$$\log \cos a = 9.4900 - 10$$

$$\log \sin B = \frac{9.7457 - 10}{}$$

$$\log \cos A = \frac{9.2357 - 10}{}$$

$$180^\circ - A = 80^\circ 55'.$$

$$A = 99^\circ 54.5'.$$

$$\log \sin a = 9.9782 - 10$$

$$\log \tan B = \frac{9.8263 - 10}{}$$

$$\log \tan b = \frac{9.8045 - 10}{}$$

$$b = 32^\circ 31.1'.$$

$$\log \tan a = 0.4882$$

$$\log \cos B = \frac{9.9194 - 10}{}$$

$$\log \tan c = \frac{0.5688}{}$$

$$180^\circ - c = 74^\circ 53.8'.$$

$$c = 105^\circ 6.2'.$$

Check.

$$\log \tan b = 9.8045 - 10$$

$$\log \tan c = \frac{0.5688}{}$$

$$\log \cos A = \frac{9.2357 - 10}{}$$

2. Given $c = 70^\circ 30'$, $A = 100^\circ$; find a , b , and B .

In this case, the three formulæ are,

$$\sin A = \frac{\sin a}{\sin c}, \quad \cos A = \frac{\tan b}{\tan c}, \quad \cos c = \cot A \cot B.$$

That is, $\sin a = \sin c \sin A$, $\tan b = \tan c \cos A$, $\cot B = \cos c \tan A$.

Here, the side a is determined from its sine; but the ambiguity is removed by the principles of § 121; for a and A must be in the same quadrant. Therefore a is obtuse, and the supplement of the angle obtained from the table must be taken.

By § 133, the check formula is

$$\tan B = \frac{\tan b}{\sin a}, \quad \text{or} \quad \sin a = \tan b \cot B.$$

Note 2. The check formula should always be expressed in terms of the functions used in determining the required parts; thus, in the case above, the check formula is transformed so as to involve $\cot B$ instead of $\tan B$.

$$\log \sin c = 9.9743 - 10$$

$$\log \sin A = \frac{9.9934 - 10}{}$$

$$\log \sin a = \frac{9.9677 - 10}{}$$

$$180^\circ - a = 68^\circ 10'.$$

$$a = 111^\circ 50'.$$

$$\log \tan c = 0.4509$$

$$\log \cos A = \frac{9.2397 - 10}{}$$

$$\log \tan b = \frac{9.6906 - 10}{}$$

$$180^\circ - b = 26^\circ 7.5'.$$

$$b = 153^\circ 52.5'.$$

$$\log \cos c = 9.5235 - 10$$

$$\log \tan A = \frac{0.7537}{}$$

$$\log \cot B = \frac{0.2772}{}$$

$$180^\circ - B = 27^\circ 50.6'.$$

$$B = 152^\circ 9.4'.$$

Check.

$$\log \tan b = 9.6906 - 10$$

$$\log \cot B = \frac{0.2772}{}$$

$$\log \sin a = \frac{9.9678 - 10}{}$$

Note 3. We observe here a difference of .0001 in the two values of $\log \sin a$. This does not necessarily indicate an error in the work, for such a small difference might easily be due to the fact that the logarithms are only *approximately* correct to the fourth decimal place.

3. Given $a = 132^\circ 6'$, $b = 77^\circ 51'$; find A , B , and c .

In this case, the three formulæ are,

$$\tan A = \frac{\tan a}{\sin b}, \quad \tan B = \frac{\tan b}{\sin a}, \quad \cos c = \cos a \cos b.$$

The check formula is

$$\cos c = \cot A \cot B, \text{ or } \cos c \tan A \tan B = 1.$$

That is, $\log \cos c + \log \tan A + \log \tan B = \log 1 = 0$.

$$\log \tan a = 0.0440$$

$$\log \cos a = 9.8263 - 10$$

$$\log \sin b = \underline{9.9901 - 10}$$

$$\log \cos b = \underline{9.3232 - 10}$$

$$\log \tan A = 0.0539$$

$$\log \cos c = \underline{9.1495 - 10}$$

$$180^\circ - A = 48^\circ 32.8'.$$

$$180^\circ - c = 81^\circ 53.4'.$$

$$A = 131^\circ 27.2'$$

$$c = 98^\circ 6.6'.$$

$$\log \tan b = 0.6670$$

Check.

$$\log \sin a = \underline{9.8704 - 10}$$

$$\log \cos c = 9.1495 - 10$$

$$\log \tan B = \underline{0.7966}$$

$$\log \tan A = 0.0539$$

$$B = 80^\circ 55.4'.$$

$$\log \tan B = \underline{0.7966}$$

$$\log 1 = 0.0000$$

4. Given $A = 105^\circ 59'$, $a = 128^\circ 33'$; find b , B , and c .

The formulæ are,

$$\sin b = \frac{\tan a}{\tan A}, \quad \sin B = \frac{\cos a}{\cos A}, \quad \sin c = \frac{\sin a}{\sin A}.$$

The check formula is $\sin B = \frac{\sin b}{\sin c}$.

In this example, each required part is determined from its sine; and as the ambiguity cannot be removed by § 121, both the acute angle obtained from the tables and its supplement must be retained in each case.

$$\log \tan a = 0.0986$$

$$\log \sin a = 9.8932 - 10$$

$$\log \tan A = \underline{0.5430}$$

$$\log \sin A = \underline{9.9828 - 10}$$

$$\log \sin b = 9.5556 - 10$$

$$\log \sin c = \underline{9.9104 - 10}$$

$$b = 21^\circ 3.9',$$

$$c = 54^\circ 26.7',$$

$$\text{or } 158^\circ 56.1'.$$

$$\text{or } 125^\circ 33.3'.$$

$$\log \cos A = 9.4399 - 10$$

Check.

$$\log \cos a = \underline{9.7946 - 10}$$

$$\log \sin b = 9.5556 - 10$$

$$\log \sin B = 9.6453 - 10$$

$$\log \sin c = \underline{9.9104 - 10}$$

$$B = 26^\circ 13.5',$$

$$\log \sin B = 9.6452 - 10$$

$$\text{or } 153^\circ 46.5'.$$

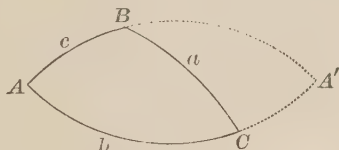
It does not follow, however, that these values can be combined promiscuously; for by § 121, since a is $> 90^\circ$, with the value of b less than 90° must be taken the value of c greater than 90° , and the value of B less than 90° ; while with the value of b greater than 90° must be taken the value of c less than 90° , and the value of B greater than 90° .

Thus the only solutions are:

$$1. \quad b = 21^\circ 3.9', \quad c = 125^\circ 33.3', \quad B = 26^\circ 13.5'.$$

$$2. \quad b = 158^\circ 56.1', \quad c = 54^\circ 26.7', \quad B = 153^\circ 46.5'.$$

Note 4. The figure shows geometrically why there are two solutions in this case.



For if AB and AC be produced to A' , forming lune $ABA'C$, triangle $A'BC$ has side a and angle A' equal, respectively, to side a and angle A of triangle ABC , and both triangles are right-angled at C .

It is evident that sides $A'B$ and $A'C$ and angle $A'BC$ are the supplements of sides c and b and angle ABC , respectively.

Solve the following right spherical triangles:

$$5. \quad \text{Given } a = 31^\circ, \quad c = 60^\circ.$$

$$6. \quad \text{Given } A = 27^\circ, \quad B = 73^\circ.$$

7. Given $a = 8^\circ$, $b = 22^\circ$.
8. Given $B = 25^\circ$, $c = 34^\circ$.
9. Given $A = 40^\circ$, $a = 26^\circ$.
10. Given $B = 127^\circ 20'$, $a = 82^\circ$.
11. Given $b = 18^\circ$, $c = 112^\circ 10'$.
12. Given $A = 120^\circ$, $c = 161^\circ 50'$.
13. Given $A = 159^\circ 40'$, $b = 135^\circ$.
14. Given $B = 110^\circ 50'$, $b = 118^\circ 30'$.
15. Given $a = 49^\circ 10'$, $b = 100^\circ$.
16. Given $A = 170^\circ 50'$, $b = 55^\circ$.
17. Given $A = 28^\circ 20'$, $c = 108^\circ 40'$.
18. Given $A = 104^\circ 50'$, $B = 156^\circ 30'$.
19. Given $a = 164^\circ 10'$, $c = 133^\circ 50'$.
20. Given $B = 99^\circ 40'$, $c = 50^\circ 30'$.
21. Given $b = 130^\circ 40'$, $c = 70^\circ 10'$.
22. Given $a = 129^\circ 30'$, $b = 166^\circ 50'$.
23. Given $A = 24^\circ 31'$, $b = 19^\circ 9'$.
24. Given $A = 83^\circ 15'$, $a = 76^\circ 46'$.
25. Given $B = 115^\circ 22'$, $a = 145^\circ 39'$.
26. Given $b = 43^\circ 57'$, $c = 62^\circ 5'$.
27. Given $A = 81^\circ 29'$, $B = 131^\circ 51'$.
28. Given $a = 147^\circ 35'$, $c = 52^\circ 13'$.
29. Given $A = 139^\circ 4'$, $c = 63^\circ 47'$.
30. Given $B = 39^\circ 43'$, $a = 54^\circ 26'$.
31. Given $b = 153^\circ 18'$, $c = 121^\circ 54'$.
32. Given $A = 37^\circ 56'$, $b = 157^\circ 12'$.
33. Given $B = 114^\circ 38'$, $c = 168^\circ 23'$.

$$34. \text{ Given } a = 66^\circ 6', \quad c = 109^\circ 44'.$$

$$35. \text{ Given } A = 30^\circ 48', \quad c = 13^\circ 27'.$$

$$36. \text{ Given } B = 69^\circ 16', \quad a = 160^\circ 55'.$$

$$37. \text{ Given } a = 142^\circ 42', \quad b = 78^\circ 6'.$$

$$38. \text{ Given } A = 126^\circ 53', \quad B = 47^\circ 34'.$$

$$39. \text{ Given } B = 16^\circ 24', \quad c = 140^\circ 37'.$$

$$40. \text{ Given } B = 98^\circ 17', \quad b = 143^\circ 8'.$$

137. Quadrantal Triangles.

By § 119, 5, the polar triangle of a quadrantal triangle is a *right* spherical triangle.

Hence, to solve a quadrantal triangle, we have only to solve its polar triangle, and take the *supplements* of the results.

1. Given $c = 90^\circ$, $a = 67^\circ 38'$, $b = 48^\circ 50'$; find A , B , and C .

Denoting the polar triangle by $A'B'C'$, we have by § 119, 5,

$$C' = 90^\circ, \quad A' = 112^\circ 22', \quad B' = 131^\circ 10'; \text{ to find } a', \quad b', \text{ and } c'.$$

By § 132, the formulæ for the solution are

$$\cos a' = \frac{\cos A'}{\sin B'}, \quad \cos b' = \frac{\cos B'}{\sin A'}, \quad \text{and} \quad \cos^+ c' = \cot A' \cot B'.$$

The check formula is $\cos c' = \cos a' \cos b'$.

$$\log \cos A' = 9.5804 - 10$$

$$\log \cot A' = 9.6143 - 10$$

$$\log \sin B' = 9.8767 - 10$$

$$\log \cot B' = 9.9417 - 10$$

$$\log \cos a' = 9.7037 - 10$$

$$\log \cos c' = 9.5560 - 10$$

$$180^\circ - a' = 59^\circ 38.2'.$$

$$c' = 68^\circ 54.8'.$$

$$\log \cos B' = 9.8184 - 10$$

Check.

$$\log \sin A' = 9.9660 - 10$$

$$\log \cos a' = 9.7037 - 10$$

$$\log \cos b' = 9.8524 - 10$$

$$\log \cos b' = 9.8524 - 10$$

$$180^\circ - b' = 44^\circ 36.7'.$$

$$\log \cos c' = 9.5561 - 10$$

Then in the given quadrantal triangle, we have

$$A = 180^\circ - a' = 59^\circ 38.2',$$

$$B = 180^\circ - b' = 44^\circ 36.7',$$

$$C = 180^\circ - c' = 111^\circ 5.2'.$$

EXAMPLES.

Solve the following quadrantal triangles:

2. Given $A = 157^\circ$, $C = 121^\circ$.
3. Given $a = 117^\circ$, $b = 142^\circ 50'$.
4. Given $A = 43^\circ$, $B = 106^\circ$.
5. Given $b = 162^\circ 20'$, $C = 64^\circ 40'$.
6. Given $A = 30^\circ 10'$, $a = 72^\circ 30'$.
7. Given $A = 118^\circ 16'$, $b = 137^\circ 57'$.
8. Given $a = 51^\circ 34'$, $C = 25^\circ 49'$.
9. Given $B = 141^\circ 13'$, $C = 49^\circ 35'$.
10. Given $a = 17^\circ 41'$, $B = 38^\circ 24'$.
11. Given $B = 159^\circ 2'$, $b = 136^\circ 28'$.

138. Isosceles Spherical Triangles.

We know, by Geometry, that if an arc of a great circle be drawn from the vertex of an isosceles spherical triangle to the middle point of the base, it is perpendicular to the base, bisects the vertical angle, and divides the triangle into two symmetrical right spherical triangles.

By solving one of these, we can find the required parts of the given triangle.

1. Given $a = 115^\circ$, $b = 115^\circ$, $C = 71^\circ 40'$; find A , B , and c .

Denoting the elements of one of the right triangles by A , B' , C' , a' , b' , and c' , where C' is the right angle, we have

$$c' = a = 115^\circ, \text{ and } A' = \frac{1}{2} C = 35^\circ 50'.$$

We have then to find the parts a and B in this triangle.

By § 128, $\sin A' = \frac{\sin a'}{\sin c'}$, and $\cos c' = \cot A' \cot B'$.

Or, $\sin a' = \sin c' \sin A'$, and $\cot B' = \cos c' \tan A'$.

$$\log \sin c' = 9.9573 - 10$$

$$\log \cos c' = 9.6259 - 10$$

$$\log \sin A' = 9.7675 - 10$$

$$\log \tan A' = 9.8586 - 10$$

$$\log \sin a' = 9.7248 - 10$$

$$\log \cot B' = 9.4845 - 10$$

$$a' = 32^\circ 3.0'.$$

$$180^\circ - B' = 73^\circ 1.8'.$$

$$B' = 106^\circ 58.2'.$$

Then in the given isosceles triangle,

$$A = B = B' = 106^\circ 58.2', \text{ and } c = 2 a' = 64^\circ 6.0'.$$

EXAMPLES.

Solve the following isosceles spherical triangles.

2. Given $A = 27^\circ$, $B = 27^\circ$, $c = 135^\circ$.
3. Given $a = 152^\circ$, $b = 152^\circ$, $C = 68^\circ$.
4. Given $a = 112^\circ 36'$, $b = 112^\circ 36'$, $c = 123^\circ 50'$.
5. Given $A = 159^\circ 14'$, $B = 159^\circ 14'$, $a = 137^\circ 47'$.

XI. OBLIQUE SPHERICAL TRIANGLES.

GENERAL PROPERTIES OF SPHERICAL TRIANGLES.

139. *In any spherical triangle, the sines of the sides are proportional to the sines of their opposite angles.*

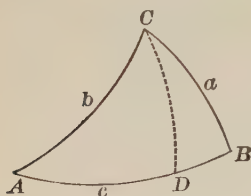


FIG. 1.

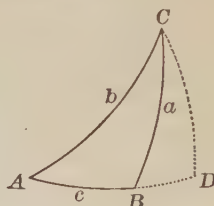


FIG. 2.

Let ABC be any spherical triangle, and draw arc CD perpendicular to AB .

There will be two cases according as CD falls upon AB (Fig. 1), or AB produced (Fig. 2).

In right spherical triangle ACD , in either figure, we have by (70),

$$\sin A = \frac{\sin CD}{\sin b}.$$

Also, in Fig. 1,
$$\sin B = \frac{\sin CD}{\sin a}.$$

And in Fig. 2,
$$\begin{aligned} \sin B &= \sin(180^\circ - CBD) \\ &= \sin CBD \text{ (§ 32)} = \frac{\sin CD}{\sin a}. \end{aligned}$$

Dividing these equations, we have in either case

$$\frac{\sin A}{\sin B} = \frac{\frac{\sin CD}{\sin b}}{\frac{\sin CD}{\sin a}} = \frac{\sin a}{\sin b}. \quad (79)$$

$$\text{In like manner, } \frac{\sin B}{\sin C} = \frac{\sin b}{\sin c}, \quad (80)$$

$$\text{and } \frac{\sin A}{\sin C} = \frac{\sin a}{\sin c}. \quad (81)$$

140. *In any spherical triangle, the cosine of any side is equal to the product of the cosines of the other two sides, plus the continued product of their sines and the cosine of their included angle.*

In right spherical triangle BCD , in Fig. 1, § 139, we have, by (69),

$$\cos a = \cos BD \cos CD = \cos (c - AD) \cos CD.$$

And in Fig. 2,

$$\cos a = \cos BD \cos CD = \cos (AD - c) \cos CD.$$

Then in either case, by (12),

$$\cos a = \cos c \cos AD \cos CD + \sin c \sin AD \cos CD.$$

But in right spherical triangle ACD , by (69),

$$\cos AD \cos CD = \cos b.$$

$$\text{Also, } \sin AD \cos CD = \sin AD \frac{\cos b}{\cos AD} = \cos b \tan AD.$$

$$\text{But, since } \tan b = \frac{\sin b}{\cos b}, \quad \cos b = \frac{\sin b}{\tan b}.$$

$$\text{Then, } \sin AD \cos CD = \sin b \frac{\tan AD}{\tan b} = \sin b \cos A, \text{ by (71).}$$

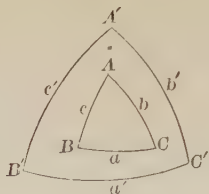
$$\text{Whence, } \cos a = \cos b \cos c + \sin b \sin c \cos A. \quad (82)$$

In like manner,

$$\cos b = \cos c \cos a + \sin c \sin a \cos B, \quad (83)$$

$$\text{and } \cos c = \cos a \cos b + \sin a \sin b \cos C. \quad (84)$$

141. Let ABC and $A'B'C'$ be a pair of polar triangles.



Applying formula (82) to triangle $A'B'C'$, we obtain

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

Putting for a' , b' , c' , and A' the values given in § 119, 5,
 $\cos (180^\circ - A) = \cos (180^\circ - B) \cos (180^\circ - C)$
 $+ \sin (180^\circ - B) \sin (180^\circ - C) \cos (180^\circ - a).$

Whence, by § 32,

$$-\cos A = (-\cos B)(-\cos C) + \sin B \sin C(-\cos a).$$

That is,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a. \quad (85)$$

Similarly,

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b, \quad (86)$$

$$\text{and} \quad \cos C = -\cos A \cos B + \sin A \sin B \cos c. \quad (87)$$

The above proof illustrates a very important application of the theory of polar triangles in Spherical Trigonometry.

If any relation has been found between the elements of a spherical triangle, an analogous relation may be derived from it, in which each side or angle is replaced by the opposite angle or side, with suitable modifications in the algebraic signs.

142. *To express the sines, cosines, and tangents of the half angles of a spherical triangle in terms of the sides of the triangle.*

From (82), § 140,

$$\sin b \sin c \cos A = \cos a - \cos b \cos c.$$

$$\text{Whence,} \quad \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \quad (\text{A})$$

Subtracting both members from 1, we have

$$\begin{aligned} 1 - \cos A &= 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c} \\ &= \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}. \end{aligned}$$

Whence, by (28), *Page 33*

$$2 \sin^2 \frac{1}{2} A = \frac{\cos(b-c) - \cos a}{\sin b \sin c}. \quad (12) \text{ Page 30}$$

But by (20), *Page 32*

$$\cos y - \cos x = 2 \sin \frac{1}{2} (x+y) \sin \frac{1}{2} (x-y). \quad (\text{B})$$

Whence,

$$2 \sin^2 \frac{1}{2} A = \frac{2 \sin \frac{1}{2} [a + (b-c)] \sin \frac{1}{2} [a - (b-c)]}{\sin b \sin c},$$

$$\text{or} \quad \sin^2 \frac{1}{2} A = \frac{\sin \frac{1}{2} (a+b-c) \sin \frac{1}{2} (a-b+c)}{\sin b \sin c}.$$

Denoting the sum of the sides, $a+b+c$, by $2s$, we have

$$a+b-c = (a+b+c) - 2c = 2s - 2c = 2(s-c),$$

$$\text{and } a-b+c = (a+b+c) - 2b = 2s - 2b = 2(s-b).$$

$$\text{Whence,} \quad \sin^2 \frac{1}{2} A = \frac{\sin(s-b) \sin(s-c)}{\sin b \sin c}.$$

$$\text{Or,} \quad \sin \frac{1}{2} A = \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin b \sin c}}. \quad (88)$$

$$\text{In like manner, } \sin \frac{1}{2} B = \sqrt{\frac{\sin(s-c) \sin(s-a)}{\sin c \sin a}}, \quad (89)$$

$$\text{and} \quad \sin \frac{1}{2} C = \sqrt{\frac{\sin(s-a) \sin(s-b)}{\sin a \sin b}}. \quad (90)$$

Again, adding both members of (A) to 1, we have

$$\begin{aligned} 1 + \cos A &= 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c} \\ &= \frac{\cos a - (\cos b \cos c - \sin b \sin c)}{\sin b \sin c}. \end{aligned}$$

Whence, by (29),

$$\begin{aligned} 2 \cos^2 \frac{1}{2} A &= \frac{\cos a - \cos(b+c)}{\sin b \sin c} \leftarrow (10) \text{ page 29} \\ &= \frac{2 \sin \frac{1}{2}(b+c+a) \sin \frac{1}{2}(b+c-a)}{\sin b \sin c}, \text{ by (B).} \end{aligned}$$

Putting $a+b+c=2s$, whence $b+c-a=2(s-a)$,

$$\cos^2 \frac{1}{2} A = \frac{\sin s \sin(s-a)}{\sin b \sin c}.$$

$$\text{Or,} \quad \cos \frac{1}{2} A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}. \quad (91)$$

$$\text{In like manner,} \quad \cos \frac{1}{2} B = \sqrt{\frac{\sin s \sin(s-b)}{\sin c \sin a}}, \quad (92)$$

$$\text{and} \quad \cos \frac{1}{2} C = \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}}. \quad (93)$$

Dividing (88) by (91), we have

$$\frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} = \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin b \sin c}} \sqrt{\frac{\sin b \sin c}{\sin s \sin(s-a)}}.$$

$$\text{Whence,} \quad \tan \frac{1}{2} A = \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin s \sin(s-a)}}. \quad (94)$$

$$\text{In like manner,} \quad \tan \frac{1}{2} B = \sqrt{\frac{\sin(s-c) \sin(s-a)}{\sin s \sin(s-b)}}, \quad (95)$$

$$\text{and} \quad \tan \frac{1}{2} C = \sqrt{\frac{\sin(s-a) \sin(s-b)}{\sin s \sin(s-c)}}. \quad (96)$$

143. To express the sines, cosines, and tangents of the half-sides of a spherical triangle in terms of the angles of the triangle.

From (85), § 141, $\sin B \sin C \cos a = \cos A + \cos B \cos C$.

$$\text{Whence, } \cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}. \quad (\text{A})$$

$$\text{Then, } 1 - \cos a = 1 - \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

$$\begin{aligned} \text{Or, } 2 \sin^2 \frac{1}{2} a &= \frac{-(\cos B \cos C - \sin B \sin C) - \cos A}{\sin B \sin C} \\ &= -\frac{\cos(B+C) + \cos A}{\sin B \sin C}. \end{aligned}$$

Then by (19), *page 31*

$$2 \sin^2 \frac{1}{2} a = -\frac{2 \cos \frac{1}{2}(B+C+A) \cos \frac{1}{2}(B+C-A)}{\sin B \sin C}.$$

Denoting the sum of the angles, $A+B+C$, by $2S$, we have $B+C-A=2(S-A)$.

$$\text{Whence, } \sin^2 \frac{1}{2} a = -\frac{\cos S \cos(S-A)}{\sin B \sin C}.$$

$$\text{Or, } \sin \frac{1}{2} a = \sqrt{-\frac{\cos S \cos(S-A)}{\sin B \sin C}}. \quad (97)$$

$$\text{In like manner, } \sin \frac{1}{2} b = \sqrt{-\frac{\cos S \cos(S-B)}{\sin C \sin A}}, \quad (98)$$

$$\text{and } \sin \frac{1}{2} c = \sqrt{-\frac{\cos S \cos(S-C)}{\sin A \sin B}}. \quad (99)$$

Again, adding both members of (A) to 1, we have

$$1 + \cos a = 1 + \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$

$$1 + \cos a = \frac{\cos A + (\cos B \cos C + \sin B \sin C)}{\sin B \sin C} \leftarrow (12) / \text{page 2}$$

$$\begin{aligned} \text{Then, } 2 \cos^2 \frac{1}{2} a &= \frac{\cos A + \cos (B - C)}{\sin B \sin C} \\ &= \frac{2 \cos \frac{1}{2} [A + B - C] \cos \frac{1}{2} [A - (B - C)]}{\sin B \sin C}. \end{aligned}$$

$$\text{Or, } \cos^2 \frac{1}{2} a = \frac{\cos \frac{1}{2} (A + B - C) \cos \frac{1}{2} (A - B + C)}{\sin B \sin C}.$$

But $A + B - C = 2(S - C)$, and $A - B + C = 2(S - B)$.

$$\text{Whence, } \cos^2 \frac{1}{2} a = \frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}.$$

$$\text{Or, } \cos \frac{1}{2} a = \sqrt{\frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}}. \quad (100)$$

In like manner,

$$\cos \frac{1}{2} b = \sqrt{\frac{\cos (S - C) \cos (S - A)}{\sin C \sin A}}, \quad (101)$$

$$\text{and } \cos \frac{1}{2} c = \sqrt{\frac{\cos (S - A) \cos (S - B)}{\sin A \sin B}}. \quad (102)$$

Dividing (97) by (100), we have

$$\tan \frac{1}{2} a = \sqrt{-\frac{\cos S \cos (S - A)}{\cos (S - B) \cos (S - C)}}. \quad (103)$$

In like manner,

$$\tan \frac{1}{2} b = \sqrt{-\frac{\cos S \cos (S - B)}{\cos (S - C) \cos (S - A)}}, \quad (104)$$

$$\text{and } \tan \frac{1}{2} c = \sqrt{-\frac{\cos S \cos (S - C)}{\cos (S - A) \cos (S - B)}}. \quad (105)$$

NAPIER'S ANALOGIES.

144. Dividing (94) by (95), we have

$$\frac{\tan \frac{1}{2} A}{\tan \frac{1}{2} B} = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}} \sqrt{\frac{\sin s \sin (s - b)}{\sin (s - c) \sin (s - a)}}.$$

$$\text{Or, } \frac{\sin \frac{1}{2} A \cos \frac{1}{2} B}{\cos \frac{1}{2} A \sin \frac{1}{2} B} = \sqrt{\frac{\sin^2 (s - b)}{\sin^2 (s - a)}} = \frac{\sin (s - b)}{\sin (s - a)}.$$

Whence by composition and division,

$$\frac{\sin \frac{1}{2} A \cos \frac{1}{2} B + \cos \frac{1}{2} A \sin \frac{1}{2} B}{\sin \frac{1}{2} A \cos \frac{1}{2} B - \cos \frac{1}{2} A \sin \frac{1}{2} B} = \frac{\sin (s-b) + \sin (s-a)}{\sin (s-b) - \sin (s-a)}.$$

Then by (9), (11), and (21), *← Page 122*

$$\frac{\sin (\frac{1}{2} A + \frac{1}{2} B)}{\sin (\frac{1}{2} A - \frac{1}{2} B)} = \frac{\tan \frac{1}{2} [s-b+s-a]}{\tan \frac{1}{2} [s-b-(s-a)]}.$$

But $s-b+s-a=2s-a-b=c.$

Whence,
$$\frac{\sin \frac{1}{2} (A+B)}{\sin \frac{1}{2} (A-B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2} (a-b)}. \quad (106)$$

145. Multiplying (94) by (95), we have

$$\tan \frac{1}{2} A \tan \frac{1}{2} B = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}} \sqrt{\frac{\sin (s-c) \sin (s-a)}{\sin s \sin (s-b)}}.$$

Or,
$$\frac{\sin \frac{1}{2} A \sin \frac{1}{2} B}{\cos \frac{1}{2} A \cos \frac{1}{2} B} = \sqrt{\frac{\sin^2 (s-c)}{\sin^2 s}} = \frac{\sin (s-c)}{\sin s}.$$

Whence by composition and division,

100
Page 27
$$\frac{\cos \frac{1}{2} A \cos \frac{1}{2} B - \sin \frac{1}{2} A \sin \frac{1}{2} B}{\cos \frac{1}{2} A \cos \frac{1}{2} B + \sin \frac{1}{2} A \sin \frac{1}{2} B} = \frac{\sin s - \sin (s-c)}{\sin s + \sin (s-c)}. \quad \left. \right\} (21) \text{ Page 20}$$

Or, by (21),

$$\frac{\cos (\frac{1}{2} A + \frac{1}{2} B)}{\cos (\frac{1}{2} A - \frac{1}{2} B)} = \frac{\tan \frac{1}{2} [s-(s-c)]}{\tan \frac{1}{2} [s+s-c]}.$$

But $s+s-c=2s-c=a+b.$

Whence,
$$\frac{\cos \frac{1}{2} (A+B)}{\cos \frac{1}{2} (A-B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2} (a+b)}. \quad (107)$$

146. Applying formula (106) to triangle $A'B'C'$, in the figure of § 141, we obtain

$$\frac{\sin \frac{1}{2} (A' + B')}{\sin \frac{1}{2} (A' - B')} = \frac{\tan \frac{1}{2} c'}{\tan \frac{1}{2} (a' - b')}.$$

But,

$$\frac{1}{2}(A' + B') = \frac{1}{2}(180^\circ - a + 180^\circ - b) = 180^\circ - \frac{1}{2}(a + b);$$

$$\frac{1}{2}(A' - B') = \frac{1}{2}(180^\circ - a - 180^\circ + b) = -\frac{1}{2}(a - b);$$

$$\frac{1}{2}c' = \frac{1}{2}(180^\circ - C) = 90^\circ - \frac{1}{2}C;$$

$$\text{and } \frac{1}{2}(a' - b') = \frac{1}{2}(180^\circ - A - 180^\circ + B) = -\frac{1}{2}(A - B).$$

Whence,

$$\frac{\sin [180^\circ - \frac{1}{2}(a + b)]}{\sin [-\frac{1}{2}(a - b)]} = \frac{\tan (90^\circ - \frac{1}{2}C)}{\tan [-\frac{1}{2}(A - B)]}.$$

Therefore, by §§ 28, 31, and 32,

$$\frac{\sin \frac{1}{2}(a + b)}{-\sin \frac{1}{2}(a - b)} = \frac{\cot \frac{1}{2}C}{-\tan \frac{1}{2}(A - B)}.$$

$$\text{Or,} \quad \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)} = \frac{\cot \frac{1}{2}C}{\tan \frac{1}{2}(A - B)}. \quad (108)$$

In like manner, from (107), we obtain

$$\frac{\cos \frac{1}{2}(A' + B')}{\cos \frac{1}{2}(A' - B')} = \frac{\tan \frac{1}{2}c'}{\tan \frac{1}{2}(a' + b')}.$$

But,

$$\frac{1}{2}(a' + b') = \frac{1}{2}(180^\circ - A + 180^\circ - B) = 180^\circ - \frac{1}{2}(A + B).$$

Whence,

$$\frac{\cos [180^\circ - \frac{1}{2}(a + b)]}{\cos [-\frac{1}{2}(a - b)]} = \frac{\tan (90^\circ - \frac{1}{2}C)}{\tan [180^\circ - \frac{1}{2}(A + B)]}.$$

Therefore, by §§ 28, 31, and 32,

$$\frac{-\cos \frac{1}{2}(a + b)}{\cos \frac{1}{2}(a - b)} = \frac{\cot \frac{1}{2}C}{-\tan \frac{1}{2}(A + B)}.$$

$$\text{Or,} \quad \frac{\cos \frac{1}{2}(a + b)}{\cos \frac{1}{2}(a - b)} = \frac{\cot \frac{1}{2}C}{\tan \frac{1}{2}(A + B)}. \quad (109)$$

147. The formulæ exemplified in §§ 144, 145, and 146 are known as *Napier's Analogies*. In each case there may be other forms according as other elements are used.

SOLUTION OF OBLIQUE SPHERICAL TRIANGLES.

148. In the solution of oblique spherical triangles, we may distinguish six cases:

1. *Given a side and the adjacent angles.*
2. *Given two sides and their included angle.*
3. *Given the three sides.*
4. *Given the three angles.*
5. *Given two sides and the angle opposite to one of them.*
6. *Given two angles and the side opposite to one of them.*

By application of the principles of § 119, 5, the solution of an example under Case 2, 4, or 6, may be made to depend upon the solution of an example under Case 1, 3, or 5, respectively; and *vice versa*.

Hence, it is not essential to consider more than *three* cases in the solution of oblique spherical triangles.

The student must carefully bear in mind the remarks made in §§ 134 and 135.

149. CASE I. *Given a side and the adjacent angles.*

1. Given $A = 70^\circ$, $B = 131^\circ 20'$, $c = 116^\circ$; find a , b , and C .

By Napier's Analogies (§§ 144, 145); we have

$$\frac{\sin \frac{1}{2}(B + A)}{\sin \frac{1}{2}(B - A)} = \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(b - a)}, \text{ and } \frac{\cos \frac{1}{2}(B + A)}{\cos \frac{1}{2}(B - A)} = \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(b + a)}.$$

$$\text{Whence, } \tan \frac{1}{2}(b - a) = \sin \frac{1}{2}(B - A) \csc \frac{1}{2}(B + A) \tan \frac{1}{2}c,$$

$$\text{and } \tan \frac{1}{2}(b + a) = \cos \frac{1}{2}(B - A) \sec \frac{1}{2}(B + A) \tan \frac{1}{2}c.$$

From the data, $\frac{1}{2}(B - A) = 30^\circ 40'$, $\frac{1}{2}(B + A) = 100^\circ 40'$, $\frac{1}{2}c = 58^\circ$.

$$\log \sin \frac{1}{2}(B - A) = 9.7076 - 10 \qquad \log \cos \frac{1}{2}(B - A) = 9.9346 - 10$$

$$\log \csc \frac{1}{2}(B + A) = 0.0076 \qquad \log \sec \frac{1}{2}(B + A) = 0.7328$$

$$\log \tan \frac{1}{2}c = 0.2042 \qquad \log \tan \frac{1}{2}c = 0.2042$$

$$\log \tan \frac{1}{2}(b - a) = 9.9194 - 10 \qquad \log \tan \frac{1}{2}(b + a) = 0.8714$$

$$\frac{1}{2}(b - a) = 39^\circ 42.8', \qquad 180^\circ - \frac{1}{2}(b + a) = 82^\circ 20.5'.$$

$$\frac{1}{2}(b + a) = 97^\circ 39.5'.$$

$$\text{Then,} \qquad a = \frac{1}{2}(b + a) - \frac{1}{2}(b - a) = 57^\circ 56.7',$$

$$\text{and} \qquad b = \frac{1}{2}(b + a) + \frac{1}{2}(b - a) = 137^\circ 22.3'.$$

To find C , we have by § 146,

$$\begin{aligned} \cot \frac{1}{2}C &= \frac{\sin \frac{1}{2}(b + a)}{\sin \frac{1}{2}(b - a)} \tan \frac{1}{2}(B - A) \\ &= \sin \frac{1}{2}(b + a) \csc \frac{1}{2}(b - a) \tan \frac{1}{2}(B - A). \end{aligned}$$

$$\log \sin \frac{1}{2}(b + a) = 9.9961 - 10$$

$$\log \csc \frac{1}{2}(b - a) = 0.1946$$

$$\log \tan \frac{1}{2}(B - A) = 9.7730 - 10$$

$$\log \cot \frac{1}{2}C = 9.9637 - 10$$

$$\frac{1}{2}C = 47^\circ 23.6', \text{ and } C = 94^\circ 47.2'.$$

Note 1. The value of C may also be determined by the formula

$$\cot \frac{1}{2}C = \frac{\cos \frac{1}{2}(b + a)}{\cos \frac{1}{2}(b - a)} \tan \frac{1}{2}(B + A) \quad (\S 146).$$

Note 2. The triangle is always possible for any values of the given elements.

EXAMPLES.

Solve the following spherical triangles:

$$2. \text{ Given } A = 87^\circ, \quad B = 61^\circ, \quad c = 112^\circ.$$

$$3. \text{ Given } B = 41^\circ, \quad C = 122^\circ, \quad a = 37^\circ.$$

$$4. \text{ Given } A = 135^\circ, \quad C = 51^\circ, \quad b = 69^\circ.$$

$$5. \text{ Given } A = 147^\circ 30', \quad B = 163^\circ 10', \quad c = 76^\circ 20'$$

(For additional examples under Case I., see § 155.)

150. CASE II. *Given two sides and their included angle.*

1. Given $b = 137^\circ 20'$, $c = 116^\circ$, $A = 70^\circ$; find B , C , and a

By Napier's Analogies (§ 146), we have

$$\frac{\sin \frac{1}{2}(b+c)}{\sin \frac{1}{2}(b-c)} = \frac{\cot \frac{1}{2}A}{\tan \frac{1}{2}(B-C)}, \text{ and } \frac{\cos \frac{1}{2}(b+c)}{\cos \frac{1}{2}(b-c)} = \frac{\cot \frac{1}{2}A}{\tan \frac{1}{2}(B+C)}.$$

$$\text{Whence, } \tan \frac{1}{2}(B-C) = \sin \frac{1}{2}(b-c) \csc \frac{1}{2}(b+c) \cot \frac{1}{2}A,$$

$$\text{and } \tan \frac{1}{2}(B+C) = \cos \frac{1}{2}(b-c) \sec \frac{1}{2}(b+c) \cot \frac{1}{2}A.$$

$$\text{From the data, } \frac{1}{2}(b-c) = 10^\circ 40', \quad \frac{1}{2}(b+c) = 126^\circ 40', \quad \frac{1}{2}A = 35^\circ.$$

$$\log \sin \frac{1}{2}(b-c) = 9.2674 - 10 \qquad \log \cos \frac{1}{2}(b-c) = 9.9924 - 10$$

$$\log \csc \frac{1}{2}(b+c) = 0.0958 \qquad \log \sec \frac{1}{2}(b+c) = 0.2239$$

$$\log \cot \frac{1}{2}A = 0.1548 \qquad \log \cot \frac{1}{2}A = 0.1548$$

$$\log \tan \frac{1}{2}(B-C) = 9.5180 - 10 \qquad \log \tan \frac{1}{2}(B+C) = 0.3711$$

$$\frac{1}{2}(B-C) = 18^\circ 14.5', \qquad 180^\circ - \frac{1}{2}(B+C) = 66^\circ 57.1'$$

$$\frac{1}{2}(B+C) = 113^\circ 2.9'.$$

$$\text{Then, } B = \frac{1}{2}(B+C) + \frac{1}{2}(B-C) = 131^\circ 17.4',$$

$$\text{and } C = \frac{1}{2}(B+C) - \frac{1}{2}(B-C) = 94^\circ 48.4'.$$

To find a , we have by § 144,

$$\tan \frac{1}{2}a = \frac{\sin \frac{1}{2}(B+C)}{\sin \frac{1}{2}(B-C)} \tan \frac{1}{2}(b-c).$$

$$\log \sin \frac{1}{2}(B+C) = 9.9639 - 10$$

$$\log \csc \frac{1}{2}(B-C) = 0.5044$$

$$\log \tan \frac{1}{2}(b-c) = 9.2750 - 10$$

$$\log \tan \frac{1}{2}a = 9.7433 - 10$$

$$\frac{1}{2}a = 28^\circ 58.3', \text{ and } a = 57^\circ 56.6'.$$

Note. The triangle is possible for any values of the given elements

EXAMPLES.

Solve the following spherical triangles:

2. Given $a = 64^\circ$, $b = 34^\circ$, $C = 48^\circ$.

3. Given $b = 42^\circ$, $c = 96^\circ$, $A = 110^\circ$

4. Given $a = 146^\circ$, $c = 69^\circ$, $B = 125^\circ$.

5. Given $a = 90^\circ 50'$, $b = 117^\circ 50'$, $C = 120^\circ$.

(For additional examples under Case II., see § 155.)

151. CASE III. *Given the three sides.*

The angles may be calculated by the formulæ of § 142.

If all the angles are to be computed, the *tangent* formulæ are the most convenient, since only four different angles occur in the second members.

If but one angle is required, the *cosine* formula involves the least work.

The triangle is possible for any values of the data, provided that no side is greater than the sum of the other two, and that the sum of the sides is less than 360° (§ 119, 1 and 2).

If all the angles are required, and the tangent formulæ are used, it is convenient to modify them as follows.

By (91),

$$\begin{aligned}\tan \frac{1}{2} A &= \sqrt{\frac{\sin(s-a) \sin(s-b) \sin(s-c)}{\sin s \sin^2(s-a)}} \\ &= \frac{1}{\sin(s-a)} \sqrt{\frac{\sin(s-a) \sin(s-b) \sin(s-c)}{\sin s}}\end{aligned}$$

Denoting $\sqrt{\frac{\sin(s-a) \sin(s-b) \sin(s-c)}{\sin s}}$ by k , we have

$$\tan \frac{1}{2} A = \frac{k}{\sin(s-a)}.$$

Similarly, $\tan \frac{1}{2} B = \frac{k}{\sin(s-b)}$, and $\tan \frac{1}{2} C = \frac{k}{\sin(s-c)}$.

1. Given $a = 57^\circ$, $b = 137^\circ$, $c = 116^\circ$; find A , B , and C .

Here,

$$2s = a + b + c = 310^\circ.$$

Whence, $s = 155^\circ$, $s - a = 98^\circ$, $s - b = 18^\circ$, $s - c = 39^\circ$.

$\log \sin(s - a) = 9.9958 - 10$ $\log \sin(s - b) = 9.4900 - 10$ $\log \sin(s - c) = 9.7989 - 10$ $\log \csc s = 0.3741$ $\begin{array}{r} 2 \overline{)19.6588 - 20} \\ \log k = 9.8294 - 10 \end{array}$ $\log \sin(s - a) = 9.9958 - 10$ $\log \tan \frac{1}{2} A = 9.8336 - 10$ $\begin{array}{r} \frac{1}{2} A = 34^\circ 17.0'. \\ A = 68^\circ 34.0'. \end{array}$	$\log k = 9.8294 - 10$ $\log \sin(s - b) = \underline{9.4900 - 10}$ $\log \tan \frac{1}{2} B = 0.3394$ $\begin{array}{r} \frac{1}{2} B = 65^\circ 24.2'. \\ B = 130^\circ 48.4'. \end{array}$ $\log k = 9.8294 - 10$ $\log \sin(s - c) = \underline{9.7989 - 10}$ $\log \tan \frac{1}{2} C = 0.0305$ $\begin{array}{r} \frac{1}{2} C = 47^\circ 0.8'. \\ C = 94^\circ 1.6'. \end{array}$
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EXAMPLES.

Solve the following spherical triangles:

2. Given $a = 69^\circ$, $b = 74^\circ$, $c = 63^\circ$.
3. Given $a = 103^\circ$, $b = 53^\circ$, $c = 61^\circ$.
4. Given $a = 91^\circ$, $b = 118^\circ$, $c = 132^\circ$.
5. Given $a = 58^\circ$, $b = 138^\circ$, $c = 116^\circ$; find A .

(For additional examples under Case III., see § 155.)

152. CASE IV. *Given the three angles.*

The sides may be calculated by the formulæ of § 143.

If all the sides are to be computed, the *tangent* formulæ are the most convenient, since only four different angles occur in the second members.

If but one angle is required, the *sine* formula involves the least work.

The triangle is possible for any values of the data, provided that the sum of the angles is between 180° and 540° (§ 119, 3), and that each of the quantities $B + C - A$, $C + A - B$, and $A + B - C$ is between 180° and -180° (§ 122).

For such values of the angles, S is between 90° and 270° , and each of the quantities $S - A$, $S - B$, and $S - C$ between 90° and -90° .

Then, $\cos S$ is $-$, while the cosines of $S - A$, $S - B$, and $S - C$ are $+$ (§ 21).

Hence, the expressions under the radical signs in the formulæ are essentially positive, and no attention need be paid to the algebraic signs.

If all the sides are required, and the tangent formulæ are used, it is convenient to modify them as follows:

By (103),

$$\begin{aligned}\tan \frac{1}{2} a &= \sqrt{-\frac{\cos S \cos^2(S - A)}{\cos(S - A) \cos(S - B) \cos(S - C)}} \\ &= \cos(S - A) \sqrt{-\frac{\cos S}{\cos(S - A) \cos(S - B) \cos(S - C)}}.\end{aligned}$$

Denoting $\sqrt{-\frac{\cos S}{\cos(S - A) \cos(S - B) \cos(S - C)}}$ by K ,

$$\tan \frac{1}{2} a = K \cos(S - A).$$

In like manner,

$$\tan \frac{1}{2} b = K \cos(S - B), \text{ and } \tan \frac{1}{2} c = K \cos(S - C).$$

1. Given $A = 150^\circ$, $B = 131^\circ$, $C = 115^\circ$; find a , b , and c .

Here, $2S = A + B + C = 396^\circ$.

Whence, $S = 198^\circ$, $S - A = 48^\circ$, $S - B = 67^\circ$, $S - C = 83^\circ$.

$\log \cos S = 9.9782 - 10$	$\log K = 0.7375$
$\log \sec(S - A) = 0.1745$	$\log \cos(S - B) = 9.5919 - 10$
$\log \sec(S - B) = 0.4081$	$\log \tan \frac{1}{2} b = 0.3294$
$\log \sec(S - C) = 0.9141$	$\frac{1}{2} b = 64^\circ 54.2'$
$2 \overline{) 1.4749}$	$b = 129^\circ 48.4'$
$\log K = 0.7375$	$\log K = 0.7375$
$\log \cos(S - A) = 9.8255 - 10$	$\log \cos(S - C) = 9.0859 - 10$
$\log \tan \frac{1}{2} a = 0.5630$	$\log \tan \frac{1}{2} c = 9.8234 - 10$
$\frac{1}{2} a = 74^\circ 42.2'$	$\frac{1}{2} c = 33^\circ 39.6'$
$a = 149^\circ 24.4'$	$c = 67^\circ 19.2'$

Note 1. By § 34, $\cos 198^\circ = -\sin 108^\circ = -\cos 18^\circ$; whence, without regard to algebraic sign, $\log \cos 198^\circ = \log \cos 18^\circ$.

2. Given $A = 123^\circ$, $B = 45^\circ$, $C = 58^\circ$; find a .

$$\text{By (97)} \quad \sin \frac{1}{2}a = \sqrt{-\frac{\cos S \cos (S-A)}{\sin B \sin C}}.$$

Here,

$$2S = A + B + C = 226^\circ; \text{ whence, } S = 113^\circ, \text{ and } S - A = -10^\circ.$$

$$\log \cos S = 9.5919 - 10$$

$$\log \cos (S - A) = 9.9934 - 10$$

$$\log \csc B = 0.1505$$

$$\log \csc C = 0.0716$$

$$2)19.8074 - 10$$

$$\log \sin \frac{1}{2}a = 9.9037 - 10$$

$$\frac{1}{2}a = 53^\circ 14.4', \text{ and } a = 106^\circ 28.8'.$$

Note 2. By § 28, $\cos (-10^\circ) = \cos 10^\circ$.

EXAMPLES.

Solve the following spherical triangles:

3. Given $A = 52^\circ$, $B = 59^\circ$, $C = 83^\circ$.

4. Given $A = 143^\circ$, $B = 28^\circ$, $C = 32^\circ$.

5. Given $A = 142^\circ$, $B = 159^\circ$, $C = 133^\circ$.

6. Given $A = 70^\circ$, $B = 122^\circ$, $C = 95^\circ$; find b .

(For additional examples under Case IV., see § 155.)

153. CASE V. *Given two sides and the angle opposite to one of them.*

1. Given $a = 58^\circ$, $b = 138^\circ$, $B = 134^\circ 50'$; find A , C , and c .

$$\text{By (79), } \frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}, \text{ or } \sin A = \sin a \csc b \sin B.$$

$$\log \sin a = 9.9284 - 10$$

$$\log \csc b = 0.1745$$

$$\log \sin B = 9.8507 - 10$$

$$\log \sin A = 9.9536 - 10$$

$$A = 63^\circ 58.6' \text{ or } 116^\circ 1.4' (\S 135).$$

To find C and c , we have by §§ 144 and 146,

$$\cot \frac{1}{2} C = \sin \frac{1}{2}(b + a) \csc \frac{1}{2}(b - a) \tan \frac{1}{2}(B - A),$$

and $\tan \frac{1}{2} c = \sin \frac{1}{2}(B + A) \csc \frac{1}{2}(B - A) \tan \frac{1}{2}(b - a).$

Using the first value of A ,

$$\frac{1}{2}(B + A) = 99^\circ 24.3', \text{ and } \frac{1}{2}(B - A) = 35^\circ 25.7'.$$

Also, $\frac{1}{2}(b + a) = 98^\circ$, and $\frac{1}{2}(b - a) = 40^\circ$.

$$\log \sin \frac{1}{2}(b + a) = 9.9958 - 10 \quad \log \sin \frac{1}{2}(B + A) = 9.9941 - 10$$

$$\log \csc \frac{1}{2}(b - a) = 0.1919 \quad \log \csc \frac{1}{2}(B - A) = 0.2368$$

$$\log \tan \frac{1}{2}(B - A) = 9.8521 - 10 \quad \log \tan \frac{1}{2}(b - a) = 9.9238 - 10$$

$$\log \cot \frac{1}{2} C = 0.0398 \quad \log \tan \frac{1}{2} c = 0.1547$$

$$\frac{1}{2} C = 42^\circ 22.7' \quad \frac{1}{2} c = 54^\circ 59.6'.$$

$$C = 84^\circ 45.4' \quad c = 109^\circ 59.2'.$$

Using the second value of A ,

$$\frac{1}{2}(B + A) = 125^\circ 25.7', \text{ and } \frac{1}{2}(B - A) = 9^\circ 24.3'.$$

$$\log \sin \frac{1}{2}(b + a) = 9.9958 - 10 \quad \log \sin \frac{1}{2}(B + A) = 9.9111 - 10$$

$$\log \csc \frac{1}{2}(b - a) = 0.1919 \quad \log \csc \frac{1}{2}(B - A) = 0.7867$$

$$\log \tan \frac{1}{2}(B - A) = 9.2192 - 10 \quad \log \tan \frac{1}{2}(b - a) = 9.9238 - 10$$

$$\log \cot \frac{1}{2} C = 9.4069 - 10 \quad \log \tan \frac{1}{2} c = 0.6216$$

$$\frac{1}{2} C = 75^\circ 40.9' \quad \frac{1}{2} c = 76^\circ 33.6'.$$

$$C = 151^\circ 21.8' \quad c = 153^\circ 7.2'.$$

Thus the two solutions are :

$$1. \ A = 63^\circ 58.6', \ C = 84^\circ 45.4', \ c = 109^\circ 59.2'.$$

$$2. \ A = 116^\circ 1.4', \ C = 151^\circ 21.8', \ c = 153^\circ 7.2'.$$

As in the corresponding case in the solution of plane oblique triangles (compare §§ 108 and 109), there may sometimes be two solutions, sometimes only one, and sometimes none, in an example under Case V.

After the two values of A have been obtained, the number of solutions may be determined by inspection; for, by § 119, 6, if a is $< b$, A must be $< B$; and if a is $> b$, A must be $> B$.

Hence, *only those values of A can be retained which are greater or less than B according as a is greater or less than b .*

Thus, in Ex. 1, a is given $< b$; and since both values of A are $< B$, we have two solutions.

Again, if the data are such as to make $\log \sin A$ positive, there will be no solution corresponding.

2. Given $a = 58^\circ$, $c = 116^\circ$, $C = 94^\circ 50'$; find A .

In this case, $\frac{\sin A}{\sin C} = \frac{\sin a}{\sin c}$, or $\sin A = \sin a \csc c \sin C$.

$$\log \sin a = 9.9284 - 10$$

$$\log \csc c = 0.0463$$

$$\log \sin C = 9.9985 - 10$$

$$\log \sin A = 9.9732 - 10$$

$$A = 70^\circ 5.0' \text{ or } 109^\circ 55.0'.$$

Since a is given $< c$, only values of A which are $< C$ can be retained; then the only solution is $A = 70^\circ 5.0'$.

3. Given $b = 126^\circ$, $c = 70^\circ$, $B = 57^\circ$; find C .

In this case, $\frac{\sin C}{\sin B} = \frac{\sin c}{\sin b}$, or $\sin C = \sin c \csc b \sin B$.

$$\log \sin c = 9.9730 - 10$$

$$\log \csc b = 0.0920$$

$$\log \sin B = 9.9236 - 10$$

$$\log \sin C = 9.9886 - 10$$

$$C = 76^\circ 56.7' \text{ or } 103^\circ 3.3'.$$

Since both values of C are $> B$, while c is given $< b$, there is no solution.

EXAMPLES.

Solve the following spherical triangles:

4. Given $a = 29^\circ$, $b = 14^\circ$, $A = 49^\circ$.

5. Given $a = 98^\circ$, $c = 36^\circ$, $C = 163^\circ$.

6. Given $b = 132^\circ$, $c = 56^\circ$, $B = 116^\circ 18'$.

7. Given $a = 104^\circ 50'$, $c = 153^\circ 20'$, $A = 70^\circ$.

8. Given $a = 111^\circ 20'$, $b = 41^\circ 40'$, $B = 25^\circ$.

(For additional examples under Case V., see § 155.)

154. CASE VI. *Given two angles and the side opposite to one of them.*

1. Given $A = 110^\circ$, $B = 131^\circ 20'$, $b = 137^\circ 20'$; find a , c , and C .

In this case, $\frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}$, or $\sin a = \sin A \csc B \sin b$.

$$\log \sin A = 9.9730 - 10$$

$$\log \csc B = 0.1244$$

$$\log \sin b = 9.8311 - 10$$

$$\log \sin a = 9.9285 - 10$$

$$a = 58^\circ 1.2' \text{ or } 121^\circ 58.8'.$$

To find c and C , we have by §§ 144 and 146,

$$\tan \frac{1}{2} c = \sin \frac{1}{2} (B + A) \csc \frac{1}{2} (B - A) \tan \frac{1}{2} (b - a),$$

and $\cot \frac{1}{2} C = \sin \frac{1}{2} (b + a) \csc \frac{1}{2} (b - a) \tan \frac{1}{2} (B - A).$

Using the first value of a ,

$$\frac{1}{2} (b + a) = 97^\circ 40.6', \text{ and } \frac{1}{2} (b - a) = 39^\circ 39.4'.$$

Also, $\frac{1}{2} (B + A) = 120^\circ 40'$, and $\frac{1}{2} (B - A) = 10^\circ 40'$.

$$\log \sin \frac{1}{2} (B + A) = 9.9346 - 10 \qquad \log \sin \frac{1}{2} (b + a) = 9.9961 - 10$$

$$\log \csc \frac{1}{2} (B - A) = 0.7326 \qquad \log \csc \frac{1}{2} (b - a) = 0.1951$$

$$\log \tan \frac{1}{2} (b - a) = 9.9185 - 10 \qquad \log \tan \frac{1}{2} (B - A) = 9.2750 - 10$$

$$\log \tan \frac{1}{2} c = 0.5857 \qquad \log \cot \frac{1}{2} C = 9.4662 - 10$$

$$\frac{1}{2} c = 75^\circ 26.9'.$$

$$\frac{1}{2} C = 73^\circ 41.5'.$$

$$c = 150^\circ 53.8'.$$

$$C = 147^\circ 23.0'.$$

Using the second value of a ,

$$\frac{1}{2} (b + a) = 129^\circ 39.4', \text{ and } \frac{1}{2} (b - a) = 7^\circ 40.6'.$$

$$\log \sin \frac{1}{2}(B + A) = 9.9346 - 10$$

$$\log \csc \frac{1}{2}(B - A) = 0.7326$$

$$\log \tan \frac{1}{2}(b - a) = 9.1297 - 10$$

$$\log \tan \frac{1}{2}c = 9.7969 - 10$$

$$\frac{1}{2}c = 32^\circ 3.9'.$$

$$c = 64^\circ 7.8'.$$

$$\log \sin \frac{1}{2}(b + a) = 9.8865 - 10$$

$$\log \csc \frac{1}{2}(b - a) = 0.8742$$

$$\log \tan \frac{1}{2}(B - A) = 9.2750 - 10$$

$$\log \cot \frac{1}{2}C = 0.0357$$

$$\frac{1}{2}C = 42^\circ 38.8'.$$

$$C = 85^\circ 17.6'.$$

Thus the two solutions are :

$$1. \ a = 58^\circ 1.2', \quad c = 150^\circ 53.8', \quad C = 147^\circ 23.0'.$$

$$2. \ a = 121^\circ 58.8', \quad c = 64^\circ 7.8', \quad C = 85^\circ 17.6'.$$

In examples in Case VI., as in Case V., there may sometimes be two solutions, sometimes only one, and sometimes none.

As in Case V., *only those values of a can be retained which are greater or less than b according as A is greater or less than B.*

Also, if $\log \sin a$ is positive, the triangle is impossible.

EXAMPLES.

Solve the following spherical triangles :

$$2. \text{ Given } A = 84^\circ, \quad B = 19^\circ, \quad a = 28^\circ.$$

$$3. \text{ Given } B = 159^\circ, \quad C = 36^\circ, \quad b = 9^\circ.$$

$$4. \text{ Given } A = 25^\circ 20', \quad C = 153^\circ 27', \quad a = 73^\circ 10'.$$

$$5. \text{ Given } A = 142^\circ 40', \quad C = 71^\circ 10', \quad c = 39^\circ 30'.$$

$$6. \text{ Given } A = 110^\circ, \quad B = 123^\circ 20', \quad b = 127^\circ.$$

(For additional examples under Case VI., see § 155.)

MISCELLANEOUS EXAMPLES.

155. Solve the following spherical triangles :

$$1. \text{ Given } a = 38^\circ, \quad b = 51^\circ, \quad c = 42^\circ.$$

$$2. \text{ Given } B = 116^\circ, \quad C = 80^\circ, \quad c = 83^\circ.$$

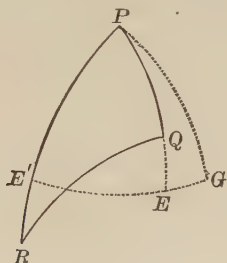
3. Given $A = 78^\circ$, $B = 41^\circ$, $c = 108^\circ$.
4. Given $b = 99^\circ 40'$, $c = 64^\circ 20'$, $B = 96^\circ 10'$.
5. Given $A = 76^\circ$, $B = 81^\circ$, $C = 61^\circ$.
6. Given $A = 62^\circ$, $C = 102^\circ$, $a = 64^\circ 30'$.
7. Given $a = 72^\circ$, $b = 47^\circ$, $C = 33^\circ$.
8. Given $A = 133^\circ 50'$, $B = 66^\circ 30'$, $a = 81^\circ 10'$.
9. Given $a = 101^\circ$, $b = 49^\circ$, $c = 60^\circ$.
10. Given $B = 135^\circ$, $C = 50^\circ$, $a = 70^\circ 20'$.
11. Given $a = 162^\circ 20'$, $b = 15^\circ 40'$, $B = 125^\circ$.
12. Given $A = 138^\circ 20'$, $B = 31^\circ 10'$, $C = 35^\circ 50'$.
13. Given $a = 109^\circ 20'$, $c = 82^\circ$, $A = 107^\circ 40'$.
14. Given $A = 132^\circ$, $B = 140^\circ$, $b = 127^\circ$.
15. Given $a = 60^\circ$, $c = 98^\circ$, $B = 110^\circ$.
16. Given $a = 55^\circ$, $c = 138^\circ 10'$, $A = 42^\circ 30'$.
17. Given $A = 61^\circ 40'$, $C = 140^\circ 20'$, $c = 150^\circ 20'$.
18. Given $a = 61^\circ$, $b = 39^\circ$, $c = 92^\circ$.
19. Given $a = 40^\circ$, $b = 118^\circ 20'$, $A = 29^\circ 20'$.
20. Given $A = 110^\circ$, $B = 131^\circ$, $C = 147^\circ$.
21. Given $a = 115^\circ 20'$, $c = 146^\circ 20'$, $C = 141^\circ 10'$.
22. Given $B = 73^\circ$, $C = 81^\circ 20'$, $b = 122^\circ 40'$.
23. Given $A = 31^\circ 40'$, $C = 122^\circ 20'$, $b = 40^\circ 40'$.
24. Given $b = 108^\circ 30'$, $c = 40^\circ 50'$, $C = 39^\circ 50'$.
25. Given $b = 120^\circ 20'$, $c = 70^\circ 40'$, $A = 50^\circ$.
26. Given $B = 22^\circ 20'$, $C = 146^\circ 40'$, $c = 138^\circ 20'$.

XII. APPLICATIONS.

156. Shortest Distance between Two Points on the Surface of the Earth.

In problems concerning navigation, the earth may be regarded as a sphere.

The *shortest distance* between any two points on the surface is the arc of a great circle which joins them; the angles between this arc and the meridians of the points determine the *bearings* of the points from each other.



Thus, if Q and R are the points, and PQ and PR their meridians, $\angle PQR$ determines the bearing of R from Q , and $\angle PRQ$ the bearing of Q from R .

If the latitudes and longitudes of Q and R are known, the arc QR and angles PQR and PRQ may be determined by the solution of a spherical triangle.

For if EE' is the equator, and PG the meridian of Greenwich,

$$\angle QPR = \angle RPG - \angle QPG = \text{longitude } R - \text{longitude } Q.$$

$$\text{Also, } PQ = PE - QE = 90^\circ - \text{latitude } Q,$$

$$\text{and } PR = PE' + RE' = 90^\circ + \text{latitude } R.$$

Thus, in spherical triangle PQR , two sides and the included angle are known, and the remaining elements may be computed.

When QR has been found in degrees, its length in miles may be calculated by finding its ratio to 360° , and multiplying the result by the length of the circumference of a great circle; in the following problems, the radius of the earth is taken as 3956 miles.

EXAMPLES.

157. In each of the following examples, find the shortest distance in miles between the places named, and the bearing of each from the other :

1. Havana (lat. $23^\circ 9' N.$, lon. $82^\circ 23' W.$), and Gibraltar (lat. $36^\circ 9' N.$, lon. $5^\circ 21' W.$).

2. Batavia (lat. $6^\circ 8' S.$, lon. $106^\circ 52' E.$), and San Francisco (lat. $37^\circ 48' N.$, lon. $122^\circ 24' W.$).

3. Vera Cruz (lat. $19^\circ 12' N.$, lon. $96^\circ 9' W.$), and Cape of Good Hope (lat. $34^\circ 22' S.$, lon. $18^\circ 29' E.$).

4. Auckland (lat. $36^\circ 51' S.$, lon. $174^\circ 50' E.$), and Callao (lat. $12^\circ 4' S.$, lon. $77^\circ 13' W.$).

5. Boston lies in lat. $42^\circ 20' N.$, lon. $71^\circ 4' W.$, and Glasgow in lat. $55^\circ 52' N.$, lon. $4^\circ 16' W.$ In what latitude does a great circle course from Boston to Glasgow cross the meridian of $40^\circ W.$?

6. Yokohama lies in lat. $35^\circ 27' N.$, lon. $139^\circ 41' E.$, and Cape Horn in lat. $55^\circ 59' S.$, lon. $67^\circ 16' W.$ In what latitude does a great circle course from Yokohama to Cape Horn cross the meridian of $160^\circ W.$?

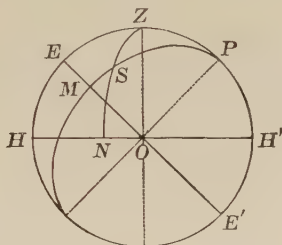
158. The Astronomical Triangle.

Let O be the position of an observer on the surface of the earth; P the celestial north-pole; Z the zenith.

The great circle EE' having P for its pole, is called the *celestial equator*; and the great circle HH' , having Z for its pole, is called the *horizon*.

Let S be the position of a star; PSM a meridian through S ; ZSN a quadrant of a great circle through Z and S .

The arc SM is called the *declination* of the star; it is called *declination north* or *south*, according as the star is north or south of the celestial equator.



The angle SPZ is called the *hour-angle* of the star; the arc SN , its *altitude*; the angle PZS , its *bearing* or *azimuth*.

The arc EZ is the *latitude* of the observer.

The spherical triangle SPZ is called the *Astronomical Triangle*.

Its sides have the following values :

$$SP = PM - SM = 90^\circ - \text{the declination of the star ;}$$

$$SZ = ZN - SN = 90^\circ - \text{the altitude of the star ;}$$

$$PZ = EP - EZ = 90^\circ - \text{the latitude of the observer.}$$

Its angle SPZ is the *hour-angle* of the star, and its angle SZP the *azimuth*.

If any three of these five elements are known, the solution of a spherical triangle serves to determine the other two.

159. Determination of Longitude and Time.

If the altitude and declination of the sun are known, and the latitude of the observer, the three sides of triangle SPZ are known, and the hour-angle SPZ may be computed.

If 24 hours be multiplied by the ratio of this angle to 360° , we obtain the time required for the sun to move from S to the meridian EP .

If this time be subtracted from 12 o'clock, if the observation is made in the morning, or added, if made in the afternoon, we obtain the *hour of the day* at the time and place of observation.

If the Greenwich time of the observation be noted on a chronometer, the difference between this and the local time as calculated above serves to determine the *longitude* of the place of observation.

In reducing time to longitude, it should be remembered that 24 hours of time correspond to 360° of longitude; that is, one hour of time corresponds to 15° of longitude, one minute to $15'$, and one second to $15''$.

EXAMPLES.

160. 1. At a certain place in latitude 40° N., the altitude of the sun was found to be 41° . If its declination at the time of the observation was 20° N., and the observation was made in the morning, how long did it take the sun to reach the meridian?

2. A mariner observes the altitude of the sun to be 60° , its declination at the hour of observation being 6° N. If the latitude of the vessel is 12° S., and the observation is made in the morning, find the hour of the day. If the observation is taken at 11.40 A.M., Greenwich time, find the longitude of the vessel.

(In this case, the side PZ of the astronomical triangle is 90° plus 12° .)

3. At what hour will the sun set in Montreal (lat. $45^\circ 30'$ N.), if its declination at sunset is 18° N.?

(At sunset, the sun's altitude is 0° , so that the side SZ of the astronomical triangle becomes 90° .)

4. A mariner observes the altitude of the sun to be $35^{\circ} 23'$, its declination being $10^{\circ} 48' \text{ S.}$ If the latitude of the vessel is $26^{\circ} 13' \text{ N.}$, and the observation is made in the afternoon, find the hour of the day. If the observation is taken at 7.13 P.M., Greenwich time, find the longitude of the vessel.

5. At what hour will the sun rise in Panama (lat. $8^{\circ} 57' \text{ N.}$), if its declination at sunrise is $23^{\circ} 2' \text{ S.}?$

6. What will be the bearing of the sun at 4 P.M. in Rio Janeiro (lat. $22^{\circ} 54' \text{ S.}$), if its declination at that time is $3^{\circ} \text{ S.}?$

7. What will be the bearing of the sun at sunrise in Boston (lat. $42^{\circ} 21' \text{ N.}$), if its declination at that time is $13^{\circ} 24' \text{ N.}?$

8. What will be the altitude of the sun at 9 A.M. in Mexico (lat. $19^{\circ} 25' \text{ N.}$), if its declination at that time is $8^{\circ} 23' \text{ N.}?$

FORMULÆ.

PLANE TRIGONOMETRY.

$$\begin{array}{lcl} \S 28. & \sin(-A) = -\sin A. & \cos(-A) = \cos A. \\ & \tan(-A) = -\tan A. & \cot(-A) = -\cot A. \\ & \sec(-A) = \sec A. & \csc(-A) = -\csc A. \end{array} \quad (1)$$

$$\begin{array}{lcl} \S 29. & \sin(90^\circ + A) = \cos A. & \cos(90^\circ + A) = -\sin A. \\ & \tan(90^\circ + A) = -\cot A. & \cot(90^\circ + A) = -\tan A. \\ & \sec(90^\circ + A) = -\csc A. & \csc(90^\circ + A) = \sec A. \end{array} \quad (2)$$

$$\begin{array}{lcl} \S 35. & \sin x = \frac{1}{\csc x}. & \tan x = \frac{1}{\cot x}. \\ & \cos x = \frac{1}{\sec x}. & \cot x = \frac{1}{\tan x}. \end{array} \quad \left. \begin{array}{l} \sec x = \frac{1}{\cos x} \\ \csc x = \frac{1}{\sin x} \end{array} \right\} (3)$$

$$\S 36. \quad \tan x = \frac{\sin x}{\cos x} \quad (4)$$

$$\S 37. \quad \cot x = \frac{\cos x}{\sin x} \quad (5)$$

$$\S 38. \quad \sin^2 x + \cos^2 x = 1. \quad (6)$$

$$\S 40. \quad \sec^2 x = 1 + \tan^2 x. \quad (7)$$

$$\csc^2 x = 1 + \cot^2 x. \quad (8)$$

$$\S 41. \quad \sin(x+y) = \sin x \cos y + \cos x \sin y. \quad (9)$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y. \quad (10)$$

$$\S 43. \quad \sin(x-y) = \sin x \cos y - \cos x \sin y. \quad (11)$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y. \quad (12)$$

$$\S 44. \quad \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \quad (13)$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \quad (14)$$

$$\cot (x+y)=\frac{\cot x \cot y-1}{\cot y+\cot x} . \quad (15)$$

$$\cot (x-y)=\frac{\cot x \cot y+1}{\cot y-\cot x} . \quad (16)$$

$$\S 45. \quad \sin x+\sin y=2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y) . \quad (17)$$

$$\sin x-\sin y=2 \cos \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y) . \quad (18)$$

$$\cos x+\cos y=2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y) . \quad (19)$$

$$\cos x-\cos y=-2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y) . \quad (20)$$

$$\S 46. \quad \frac{\sin x+\sin y}{\sin x-\sin y}=\frac{\tan \frac{1}{2}(x+y)}{\tan \frac{1}{2}(x-y)} . \quad (21)$$

§ 47.

$$\sin 2 x=2 \sin x \cos x . \quad (22) \quad \cos 2 x=\cos ^2 x-\sin ^2 x . \quad (23)$$

$$\cos 2 x=1-2 \sin ^2 x . \quad (24) \quad \cos 2 x=2 \cos ^2 x-1 . \quad (25)$$

$$\tan 2 x=\frac{2 \tan x}{1-\tan ^2 x} . \quad (26) \quad \cot 2 x=\frac{\cot ^2 x-1}{2 \cot x} . \quad (27)$$

§ 48.

$$2 \sin ^2 \frac{1}{2} x=1-\cos x . \quad (28) \quad 2 \cos ^2 \frac{1}{2} x=1+\cos x . \quad (29)$$

$$\tan \frac{1}{2} x=\frac{1-\cos x}{\sin x} . \quad (30) \quad \cot \frac{1}{2} x=\frac{1+\cos x}{\sin x} . \quad (31)$$

§ 97.

$$4 K=c^2 \sin 2 A . \quad (32) \quad 4 K=c^2 \sin 2 B . \quad (33)$$

$$2 K=a^2 \cot A . \quad (34) \quad 2 K=b^2 \cot B . \quad (35)$$

$$2 K=a^2 \tan B . \quad (36) \quad 2 K=b^2 \tan A . \quad (37)$$

$$2 K=a \sqrt{(c+a)(c-a)} . \quad (38) \quad 2 K=b \sqrt{(c+b)(c-b)} . \quad (39)$$

$$2 K=a b . \quad (40)$$

$$\S 99. \quad a: b=\sin A: \sin B . \quad (41)$$

$$b: c=\sin B: \sin C . \quad (42)$$

$$c: a=\sin C: \sin A . \quad (43)$$

$$\S 100. \quad \frac{a+b}{a-b} = \tan \frac{1}{2}(A+B). \quad (44)$$

$$\frac{b+c}{b-c} = \tan \frac{1}{2}(B+C). \quad (45)$$

$$\frac{c+a}{c-a} = \tan \frac{1}{2}(C+A). \quad (46)$$

$$\S 101. \quad a^2 = b^2 + c^2 - 2bc \cos A. \quad (47)$$

$$b^2 = c^2 + a^2 - 2ca \cos B. \quad (48)$$

$$c^2 = a^2 + b^2 - 2ab \cos C. \quad (49)$$

$$\S 102. \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}. \quad (50)$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}. \quad (51)$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}. \quad (52)$$

$$\S 103. \quad \sin \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}. \quad (53)$$

$$\sin \frac{1}{2} B = \sqrt{\frac{(s-c)(s-a)}{ca}}. \quad (54)$$

$$\sin \frac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{ab}}. \quad (55)$$

$$\cos \frac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}. \quad (56)$$

$$\cos \frac{1}{2} B = \sqrt{\frac{s(s-b)}{ca}}. \quad (57)$$

$$\cos \frac{1}{2} C = \sqrt{\frac{s(s-c)}{ab}}. \quad (58)$$

$$\tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \quad (59)$$

$$\tan \frac{1}{2} B = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}. \quad (60)$$

$$\tan \frac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}. \quad (61)$$

§ 104.

$$2 K = bc \sin A. \quad (62) \quad 2 K = \frac{a^2 \sin B \sin C}{\sin A}. \quad (65)$$

$$2 K = ca \sin B. \quad (63) \quad 2 K = \frac{b^2 \sin C \sin A}{\sin B}. \quad (66)$$

$$2 K = ab \sin C. \quad (64) \quad 2 K = \frac{c^2 \sin A \sin B}{\sin C}. \quad (67)$$

$$K = \sqrt{s(s-a)(s-b)(s-c)}. \quad (68)$$

SPHERICAL TRIGONOMETRY.

§ 123. $\cos c = \cos a \cos b. \quad (69)$

$$\sin A = \frac{\sin a}{\sin c}. \quad (70) \quad \sin B = \frac{\sin b}{\sin c}. \quad (72)$$

$$\cos A = \frac{\tan b}{\tan c}. \quad (71) \quad \cos B = \frac{\tan a}{\tan c}. \quad (73)$$

§ 124.

$$\tan A = \frac{\tan a}{\sin b}. \quad (74) \quad \tan B = \frac{\tan b}{\sin a}. \quad (75)$$

§ 125.

$$\sin A = \frac{\cos B}{\cos b}. \quad (76) \quad \sin B = \frac{\cos A}{\cos a}. \quad (77)$$

§ 126. $\cos c = \cot A \cot B. \quad (78)$

§ 139. $\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}. \quad (79)$

$$\frac{\sin B}{\sin C} = \frac{\sin b}{\sin c}. \quad (80)$$

$$\frac{\sin C}{\sin A} = \frac{\sin c}{\sin a}. \quad (81)$$

$$\S 140. \quad \cos a = \cos b \cos c + \sin b \sin c \cos A. \quad (82)$$

$$\cos b = \cos c \cos a + \sin c \sin a \cos B. \quad (83)$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \quad (84)$$

$$\S 141. \quad \cos A = -\cos B \cos C + \sin B \sin C \cos a. \quad (85)$$

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b. \quad (86)$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c. \quad (87)$$

$$\S 142. \quad \sin \frac{1}{2} A = \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin b \sin c}}. \quad (88)$$

$$\sin \frac{1}{2} B = \sqrt{\frac{\sin(s-c) \sin(s-a)}{\sin c \sin a}}. \quad (89)$$

$$\sin \frac{1}{2} C = \sqrt{\frac{\sin(s-a) \sin(s-b)}{\sin a \sin b}}. \quad (90)$$

$$\cos \frac{1}{2} A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}. \quad (91)$$

$$\cos \frac{1}{2} B = \sqrt{\frac{\sin s \sin(s-b)}{\sin c \sin a}}. \quad (92)$$

$$\cos \frac{1}{2} C = \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}}. \quad (93)$$

$$\tan \frac{1}{2} A = \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin s \sin(s-a)}}. \quad (94)$$

$$\tan \frac{1}{2} B = \sqrt{\frac{\sin(s-c) \sin(s-a)}{\sin s \sin(s-b)}}. \quad (95)$$

$$\tan \frac{1}{2} C = \sqrt{\frac{\sin(s-a) \sin(s-b)}{\sin s \sin(s-c)}}. \quad (96)$$

$$\S 143. \quad \sin \frac{1}{2} a = \sqrt{-\frac{\cos S \cos (S-A)}{\sin B \sin C}}. \quad (97)$$

$$\sin \frac{1}{2} b = \sqrt{-\frac{\cos S \cos (S-B)}{\sin C \sin A}}. \quad (98)$$

$$\sin \frac{1}{2} c = \sqrt{-\frac{\cos S \cos (S-C)}{\sin A \sin B}}. \quad (99)$$

$$\cos \frac{1}{2} a = \sqrt{\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}}. \quad (100)$$

$$\cos \frac{1}{2} b = \sqrt{\frac{\cos (S-C) \cos (S-A)}{\sin C \sin A}}. \quad (101)$$

$$\cos \frac{1}{2} c = \sqrt{\frac{\cos (S-A) \cos (S-B)}{\sin A \sin B}}. \quad (102)$$

$$\tan \frac{1}{2} a = \sqrt{-\frac{\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}}. \quad (103)$$

$$\tan \frac{1}{2} b = \sqrt{-\frac{\cos S \cos (S-B)}{\cos (S-C) \cos (S-A)}}. \quad (104)$$

$$\tan \frac{1}{2} c = \sqrt{-\frac{\cos S \cos (S-C)}{\cos (S-A) \cos (S-B)}}. \quad (105)$$

$$\S 144. \quad \frac{\sin \frac{1}{2} (A+B)}{\sin \frac{1}{2} (A-B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2} (a-b)}. \quad (106)$$

$$\S 145. \quad \frac{\cos \frac{1}{2} (A+B)}{\cos \frac{1}{2} (A-B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2} (a+b)}. \quad (107)$$

$$\S 146. \quad \frac{\sin \frac{1}{2} (a+b)}{\sin \frac{1}{2} (a-b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2} (A-B)}. \quad (108)$$

$$\frac{\cos \frac{1}{2} (a+b)}{\cos \frac{1}{2} (a-b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2} (A+B)}. \quad (109)$$

ANSWERS.

§ 54; page 40.

- | | |
|--------------------------------|--------------------------------|
| 13. $85^{\circ} 56' 37.32''$. | 15. $95^{\circ} 29' 34.8''$. |
| 14. $14^{\circ} 19' 26.22''$. | 16. $22^{\circ} 55' 5.952''$. |

§ 72 page 53.

- | | | | |
|------------|------------|-------------|-------------|
| 2. 1.5441. | 6. 2.1673. | 10. 2.4592. | 14. 3.3434. |
| 3. 1.4771. | 7. 2.3522. | 11. 2.8363. | 15. 3.8963. |
| 4. 1.9912. | 8. 2.2431. | 12. 2.7023. | 16. 3.7656. |
| 5. 1.9242. | 9. 2.6232. | 13. 2.5741. | 17. 4.1494. |

§ 74; page 54.

- | | | | |
|------------|------------|-------------|-------------|
| 2. .1549. | 5. 1.5229. | 8. .5192. | 11. 1.3734. |
| 3. .2431. | 6. .2273. | 9. .6478. | 12. .8942. |
| 4. 1.6532. | 7. 2.0212. | 10. 2.7202. | 13. 1.9842. |

§ 77; page 55.

- | | | | |
|------------|------------|------------|------------|
| 3. 2.4080. | 8. 2.2415. | 13. .2510. | 19. .9132. |
| 4. .6036. | 9. .0954. | 14. .4095. | 20. .1643. |
| 5. 1.0485. | 10. .1409. | 16. .0409. | 21. .3726. |
| 6. 8.1160. | 11. .0777. | 17. .7264. | 22. .1118. |
| 7. .4704. | 12. .3618. | 18. .1511. | 23. .8618. |

§ 81; page 57.

- | | | | |
|-----------------|-----------------|------------------|-------------|
| 2. 0.3801. | 5. 8.2831 — 10. | 8. 8.3892 — 10. | 11. 2.3043. |
| 3. 1.2252. | 6. 7.1303 — 10. | 9. 6.6865 — 10. | 12. 0.1459. |
| 4. 9.9084 — 10. | 7. 3.7693. | 10. 9.0124 — 10. | 13. 1.6505. |

§ 82; page 58.

4. 2.8878.	10. 7.6055 — 10.	18. 1.646.	24. 9.493.
5. 3.0237.	11. 6.8560 — 10.	19. 8886.	25. .2079.
6. 8.5177 — 10.	12. 0.7144.	20. 545.9.	26. 44.48.
7. 9.7164 — 10.	13. 3.0155.	21. .01461.	27. .0001109
8. 1.3028.	14. 8.9379 — 10.	22. .003318.	28. 63330.
9. 4.9659.	15. 5.0610 — 10.	23. 102.2.	29. .01301.
	30. .00005029.		

§ 87; pages 61 to 63.

1. 2.151.	16. — .002555.	31. — .3702.	48. .2985.
2. 19.38.	17. 3692.	34. 13.83.	49. .04477.
3. — 3135.	18. .2777.	35. 2.487.	50. .7945.
4. .09778.	19. — 15890.	36. 1.056.	51. 1.805.
5. .009213.	20. .03162.	37. .00002143.	52. 179.5.
6. — .1088.	21. 244.1.	38. .007105.	53. 1.883.
7. 6.359.	22. .002791.	39. .6955.	54. — 8894.
8. .03017.	23. .0000002373.	40. .5428.	55. 1.344.
9. — 3.119.	24. 2.236.	41. — 36.03.	56. — .01335.
10. 1327.	25. 1.149.	42. — 11.11.	57. 37.82.
11. 847.8.	26. — 1.220.	43. .9432.	58. .00001146
12. — .005421.	27. 1.778.	44. 2.627.	59. .00000000001782.
13. 1.205.	28. .6683.	45. 2.534.	60. 4.698.
14. .2357.	29. .6458.	46. — 1.795.	61. — .03402.
15. — 11.54.	30. .1378.	47. 1.032.	

§ 88; page 64.

1. 9.8556 — 10.	8. 9.9535 — 10.	15. 80° 26.3'.	22. .2482.
2. 9.9458 — 10.	9. 0.7654.	16. 31° 20.4'.	23. .7033.
3. 0.6518.	10. 0.0420.	17. 8° 53.5'.	24. .3886.
4. 9.9501 — 10.	11. 83° 5.2'.	18. 5° 17.6'.	25. 47° 36.3'.
5. 0.8550.	12. 33° 17.8'.	19. 40° 20.7'.	26. 21° 52.7'.
6. 9.7070 — 10.	13. 46° 40.9'.	20. 66° 43.3'.	27. 49° 57.0'.
7. 9.7547 — 10.	14. 73° 33.4'.	21. .2960.	28. 28° 46.7'.

§ 94; pages 67 to 69.

- | | |
|---|--|
| 1. $a = 1.812$, $b = 6.761$. | 17. $A = 55^\circ 44.8'$, $c = 4116$. |
| 2. $b = 12.38$, $c = 13.35$. | 18. $b = .6441$, $c = .6503$. |
| 3. $a = 16.78$, $c = 26.11$. | 19. $A = 76^\circ 34.0'$, $a = 2423$. |
| 4. $A = 34^\circ 22.2'$, $b = .5118$. | 20. $a = .2072$, $b = .4212$. |
| 5. $A = 32^\circ 44.4'$, $c = 49.92$. | 21. $a = 5091$, $c = 5268$. |
| 6. $b = 10.35$, $c = 13.14$. | 22. $b = .8478$, $c = 1.234$. |
| 7. $a = .005916$, $b = .01269$. | 23. $A = 39^\circ 22.0'$, $b = 121.2$. |
| 8. $A = 39^\circ 49.1'$, $a = 488.7$. | 24. $a = 8.243$, $c = 9.275$. |
| 9. $a = 148.4$, $c = 948.6$. | 25. $b = .000005736$, $c = .00002118$. |
| 10. $A = 49^\circ 55.0'$, $c = 4.457$. | 26. $a = .0006772$, $b = .0003899$. |
| 11. $b = 77.38$, $c = 91.08$. | 27. $A = 43^\circ 45.7'$, $b = 66650$. |
| 12. $a = 3814$, $b = 3651$. | 28. $a = 30.51$, $b = 18.59$. |
| 13. $A = 24^\circ 23.3'$, $a = .02126$. | 29. $a = 24540$, $c = 30010$. |
| 14. $a = 156.6$, $c = 856.4$. | 30. $A = 60^\circ 14.1'$, $c = .007745$. |
| 15. $a = .003607$, $b = .008830$. | 31. $c = 25.40$. |
| 16. $a = 24840$, $c = 36090$. | 32. $a = .2923$. |
| 33. $c = 949.8$. | 34. $A = 34^\circ 36.7'$. |
| 36. $a = 1.491$. | 35. $c = 4.488$. |
| | 37. $a = .03446$. |

§ 96; pages 69 to 72.

- | | | | |
|---|---|------------------------|------------------------|
| 2. 416.1 ft. | 6. 23.26. | 10. 15.27. | 14. $107^\circ 3.6'$. |
| 3. 651.8. | 7. 285.1 ft. | 11. 70.91 ft. | 15. 135.2 ft. |
| 4. 34.07. | 8. 11.695. | 12. $121^\circ 0.8'$. | 16. 5.036. |
| 5. $6^\circ 2.3'$. | 9. $52^\circ 4.2'$. | 13. $39^\circ 12.0'$. | 17. $53^\circ 8.1'$. |
| 18. Perimeter, 3.1908; diameter circumscribed circle, 1.2278. | | | |
| 19. Radius inscribed circle, 28.58; circumscribed, 30.94. | | | |
| 20. 740.2. | 21. Height of cliff, 144.4 ft.; of lighthouse, 153.6 ft | | |
| 22. 1131.3 ft. | 25. $17^\circ 1.6'$. | 28. 14.9 ft. | 31. $3^\circ 43.9'$. |
| 23. 62.9 ft. | 26. 18.63. | 29. 1575 mi. | 32. 195.9 ft. |
| 24. $12^\circ 28.9'$. | 27. 229.02. | 30. 109.0 mi. | 33. 60.14 ft. |
| 34. Bearing, S. $42^\circ 28.8'$ W.; distance, 17.77 mi. | | | |

§ 98; page 74.

- | | | | |
|-----------------|-----------|--------------|-----------|
| 2. 3.564. | 4. 13440. | 6. 46.0. | 8. .02036 |
| 3. .1098. | 5. 12.64. | 7. .0004838. | 9. 795. |
| 10. .000001323. | 11. 2840. | | |

§ 105; pages 82, 83.

- | | |
|----------------------------------|--------------------------------|
| 2. $b = 282.9$, $c = 268.5$. | 5. $a = 31.49$, $c = 49.88$. |
| 3. $a = 3.384$, $c = 9.828$. | 6. $a = .5042$, $b = .3618$. |
| 4. $a = .02893$, $b = .01825$. | 7. $b = 5499$, $c = 2959$. |

§ 106; page 84.

- | | |
|--|---|
| 2. $A = 118^\circ 18.0'$, $b = 44.73$. | 5. $B = 76^\circ 12.9'$, $c = 6.362$. |
| 3. $A = 29^\circ 59.5'$, $c = 1419$. | 6. $C = 96^\circ 3.3'$, $b = 5141$. |
| 4. $C = 88^\circ 34.8$, $a = .4038$. | 7. $B = 146^\circ 26.3'$, $a = .01044$. |

§ 107; page 86.

- | |
|---|
| 3. $A = 44^\circ 25.0'$, $B = 78^\circ 28.0'$, $C = 57^\circ 7.4'$. |
| 4. $A = 71^\circ 47.4'$, $B = 58^\circ 45.6'$, $C = 49^\circ 27.6'$. |
| 5. $A = 29^\circ 55.2'$, $B = 22^\circ 31.4'$, $C = 127^\circ 34.4'$. |
| 6. $71^\circ 33.4'$. 7. $31^\circ 6.6'$. 8. $134^\circ 29.4'$. |

§ 111; page 90.

- | | |
|--|---|
| 1. $B = 48^\circ 32.7'$, $c = 8.522$. | 6. $C = 90^\circ$, $b = 5939$. |
| 2. $B = 29^\circ 21.4'$, $a = 102.2$. | 7. $A = 14^\circ 5.4'$, $b = .1435$. |
| 3. Impossible. | 8. $A = 25^\circ 31.9'$, $c = 278.4$. |
| 4. $C = 44^\circ 56.2'$, $a = 66.68$; | 9. $C = 46^\circ 21.7'$, $b = .8728$; |
| or, $C = 135^\circ 3.8'$, $a = 12.89$. | or, $C = 133^\circ 38.3'$, $b = .2351$. |
| 5. Impossible. | |

§ 112; pages 90, 91.

- | | |
|--|---|
| 1. $A = 61^\circ 25.7'$, $c = 1018$. | 2. $a = 15.52$, $b = 10.29$. |
| 3. $A = 44^\circ 37.8'$, $B = 101^\circ 50.0'$, $C = 33^\circ 33.4'$. | |
| 4. $A = 34^\circ 25.0'$, $b = .7135$. | 6. $C = 14^\circ 57.7'$, $b = 5074$. |
| 5. $B = 46^\circ 2.1'$, $a = .05676$. | 7. $a = .0004395$, $c = .0002092$. |
| 8. $A = 51^\circ 52.6'$, $B = 42^\circ 59.0'$, $C = 85^\circ 8.8'$. | |
| 9. $B = 34^\circ 7.8'$, $c = 1223$; | 11. $C = 48^\circ 37.3'$, $a = 7.597$. |
| or, $B = 145^\circ 52.2'$, $c = 269.3$. | 12. $b = 55610$, $c = 79270$. |
| 10. $C = 46^\circ 37.8'$, $a = 7577$. | 13. Impossible. |
| 14. $A = 54^\circ 5.4'$, $b = .01073$. | |
| 15. $A = 72^\circ 43.8'$, $B = 47^\circ 40.6'$, $C = 59^\circ 36.6'$. | |
| 16. $A = 90^\circ$, $c = .1379$. | 20. $A = 27^\circ 34.2$, $c = .01875$. |
| 17. Impossible. | 21. $C = 134^\circ 36.9'$, $b = 273.3$. |
| 18. $b = .4392$, $c = .4723$. | 22. $B = 63^\circ 58.6'$, $a = 25660$; |
| 19. $B = 95^\circ 22.2'$, $c = 38.25$. | or, $B = 116^\circ 1.4'$, $a = 7573$. |

§ 113; pages 91, 92.

2. 582.	7. 24530.	12. 10.28.	17. 7255.
3. 53.8.	8. .000003186.	13. 883.2.	18. 19.19.
4. 20.98.	9. .1682.	14. 34840.	19. 47210.
5. .0780.	10. .0000002941.	15. .03519.	20. .00003759.
6. 271.3.	11. 28.77.	16. .003042.	21. .000675.

§ 114; pages 92 to 95.

- 608.4 ft.
- 420.0 sq. rd.
- 525.8 ft.
- $34^\circ 22.2'$ or $145^\circ 37.8'$.
- Height of tower, 212.8 ft.; distances, 236.4 ft., 436.4 ft.
- Distance, 21.20 mi.; bearing of first from second, N. $74^\circ 1.7'$ E.
- Opposite angles, $76^\circ 9.2'$, $57^\circ 42.2'$; remaining side, .6313.
- $B = 51^\circ 30.5'$, $c = 53.51$, $a = 46.80$.
- 1.658.
- 7.087, 11.30.
- 3995 sq. ft.
- 61.51, 58.48.
- 14.922 miles an hour.
- Height, 69.71 ft.; distance, 86.08 ft.
- 82.70 ft.
- 173.2 ft.
- 19.92, 16.62.
- One angle, $118^\circ 6.6'$; diagonal, 91.02.
- 9.012 mi.
- $AD = 9.282$, $CD = 10.65$.
- 1538 ft.
- $AD = 74.98$, $A = 68^\circ 58.3'$.
- One angle, $59^\circ 44.8'$; sides, 66.99, 37.77.
- 84.28 ft.
- 272.4 ft.
- Bluff, 438.7 ft.; lighthouse, 280.1 ft.

§ 136; pages 113 to 115.

- $A = 36^\circ 29.4'$, $B = 69^\circ 42.1'$, $b = 54^\circ 18.9'$.
- $a = 21^\circ 18.0'$, $b = 49^\circ 54.7'$, $c = 53^\circ 8.2'$.
- $A = 20^\circ 33.8'$, $B = 70^\circ 59.5'$, $c = 23^\circ 18.3'$.
- $A = 68^\circ 51.6'$, $a = 31^\circ 26.4'$, $b = 13^\circ 40.2'$.
- $B = 58^\circ 27.5'$, $b = 35^\circ 32.4'$, $c = 42^\circ 59.3'$;
- or, $B = 121^\circ 32.5'$, $b = 144^\circ 27.6'$, $c = 137^\circ 0.7'$.
- $A = 83^\circ 38.8'$, $b = 127^\circ 36.2'$, $c = 94^\circ 52.3'$.
- $A = 97^\circ 36.4'$, $a = 113^\circ 22.4'$, $B = 19^\circ 29.4'$.
- $a = 164^\circ 20'$, $B = 31^\circ 16.9'$, $b = 9^\circ 19.1'$.
- $a = 165^\circ 18.8'$, $B = 104^\circ 13.4'$, $c = 46^\circ 50.4'$.
- $A = 48^\circ 10.9'$, $a = 44^\circ 29.2'$, $c = 109^\circ 52.5'$;
- or, $A = 131^\circ 49.1'$, $a = 135^\circ 30.8'$, $c = 70^\circ 7.5'$.

15. $A = 49^\circ 35.8'$, $B = 97^\circ 36.0'$, $c = 96^\circ 31.2'$.
 16. $a = 172^\circ 28.1'$, $B = 84^\circ 45.4'$, $c = 124^\circ 39.3'$.
 17. $a = 26^\circ 42.8'$, $B = 99^\circ 47.4'$, $b = 110^\circ 59.7'$.
 18. $a = 129^\circ 56.7'$, $b = 161^\circ 32.5'$, $c = 52^\circ 28.2'$.
 19. $A = 157^\circ 46.8'$, $B = 74^\circ 12.0'$, $b = 43^\circ 56.9'$.
 20. $A = 165^\circ 0.6'$, $a = 168^\circ 29.2'$, $b = 130^\circ 28.2'$.
 21. $A = 114^\circ 49.3'$, $a = 121^\circ 22.9'$, $B = 126^\circ 14.4'$.
 22. $A = 100^\circ 38.0'$, $B = 163^\circ 8.0'$, $c = 51^\circ 44.4'$.
 23. $a = 8^\circ 30.5'$, $B = 66^\circ 55.5'$, $c = 20^\circ 53.7'$.
 24. $B = 30^\circ 53.3'$, $b = 30^\circ 12.9'$, $c = 78^\circ 35.0'$;
 or, $B = 149^\circ 6.7'$, $b = 149^\circ 47.1'$, $c = 101^\circ 25.0'$.
 25. $A = 138^\circ 15.5'$, $b = 130^\circ 2.3'$, $c = 57^\circ 55.4'$.
 26. $A = 59^\circ 17.1'$, $a = 49^\circ 26.0'$, $B = 51^\circ 46.0'$.
 27. $a = 78^\circ 32.1'$, $b = 132^\circ 25.0'$, $c = 97^\circ 42.6'$.
 28. $A = 137^\circ 17.7'$, $B = 119^\circ 29.5'$, $b = 136^\circ 31.7'$.
 29. $a = 144^\circ 0.6'$, $B = 110^\circ 57.9'$, $b = 123^\circ 6.1'$.
 30. $A = 68^\circ 10.6'$, $b = 34^\circ 3.0'$, $c = 61^\circ 11.3'$.
 31. $A = 71^\circ 45.5'$, $a = 53^\circ 44.7'$, $B = 148^\circ 2.5'$.
 32. $a = 16^\circ 48.5'$, $B = 124^\circ 31.6'$, $c = 151^\circ 56.7'$.
 33. $A = 25^\circ 4.8'$, $a = 4^\circ 53.9'$, $b = 169^\circ 27.2'$.
 34. $A = 76^\circ 16.7'$, $B = 144^\circ 1.1'$, $b = 146^\circ 26.2'$.
 35. $a = 6^\circ 50.4'$, $B = 59^\circ 54.0'$, $b = 11^\circ 36.6'$.
 36. $A = 152^\circ 7.1'$, $b = 40^\circ 48.8'$, $c = 135^\circ 39.6'$.
 37. $A = 142^\circ 5.8'$, $B = 82^\circ 43.4'$, $c = 99^\circ 26.4'$.
 38. $a = 144^\circ 24.4'$, $b = 32^\circ 28.8'$, $c = 133^\circ 19.3'$.
 39. $A = 102^\circ 48.8'$, $a = 141^\circ 46.9'$, $b = 10^\circ 19.1'$.
 40. $A = 10^\circ 22.5'$, $a = 6^\circ 16.1'$, $c = 142^\circ 41.2'$;
 or, $A = 169^\circ 37.5'$, $a = 173^\circ 43.9'$, $c = 37^\circ 18.8'$.

§ 137; page 116.

2. $a = 152^\circ 52.8'$, $b = 104^\circ 46.7'$, $B = 124^\circ 1.1'$.
 3. $A = 138^\circ 42.7'$, $B = 153^\circ 25.0'$, $C = 132^\circ 14.3'$.
 4. $a = 44^\circ 8.1'$, $b = 101^\circ 3.9'$, $C = 78^\circ 22.3'$.
 5. $a = 82^\circ 14.4'$, $A = 63^\circ 34.8'$, $B = 164^\circ 4.8'$.
 6. $b = 20^\circ 21.2'$, $B = 10^\circ 33.7'$, $C = 148^\circ 11.9'$;
 or, $b = 159^\circ 38.8'$, $B = 169^\circ 26.3'$, $C = 31^\circ 48.1'$.

7. $a = 108^\circ 29.5'$, $B = 141^\circ 31.9'$, $C = 111^\circ 45.7'$.
8. $A = 19^\circ 56.3'$, $b = 138^\circ 36.4'$, $B = 163^\circ 15.7'$.
9. $a = 46^\circ 49.3'$, $A = 33^\circ 43.3'$, $b = 124^\circ 37.8'$.
10. $A = 11^\circ 12.1'$, $b = 76^\circ 13.8'$, $C = 140^\circ 14.8'$.
11. $a = 50^\circ 56.0'$, $A = 23^\circ 47.2'$, $C = 31^\circ 17.6'$;

or, $a = 129^\circ 4.0'$, $A = 156^\circ 12.8'$, $C = 148^\circ 42.4'$.

§ 138; page 117.

2. $C = 159^\circ 59.8'$, $a = 69^\circ 44.6'$. 4. $C = 145^\circ 45.0'$, $A = 141^\circ 17.0'$.
3. $A = 120^\circ 46.6'$, $c = 30^\circ 26.6'$. 5. $C = 148^\circ 37.6'$, $c = 80^\circ 36.8'$.

§ 149; page 128.

2. $a = 98^\circ 19.9'$, $b = 60^\circ 3.9'$, $C = 110^\circ 38.6'$.
3. $c = 72^\circ 14.1'$, $b = 47^\circ 27.3'$, $A = 32^\circ 24.2'$.
4. $a = 120^\circ 34.6'$, $c = 71^\circ 7.6'$, $B = 50^\circ 3.6'$.
5. $b = 153^\circ 48.4'$, $a = 125^\circ 0.8'$, $C = 140^\circ 24.6'$.

§ 150; pages 129, 130.

2. $A = 110^\circ 46.6'$, $B = 35^\circ 34.2'$, $c = 45^\circ 36.0'$.
3. $C = 78^\circ 55.5'$, $B = 41^\circ 19.7'$, $a = 107^\circ 47.6'$.
4. $A = 145^\circ 11.8'$, $C = 107^\circ 39.8'$, $b = 126^\circ 37.4'$.
5. $B = 121^\circ 43.2'$, $A = 105^\circ 52.8'$, $c = 115^\circ 48.6'$.

§ 151; page 131.

2. $A = 74^\circ 12.4'$, $B = 82^\circ 12.0'$, $C = 66^\circ 41.4'$.
3. $A = 78^\circ 32.0'$, $B = 87^\circ 7.0'$, $C = 93^\circ 29.6'$.
4. $A = 120^\circ 21.4'$, $B = 130^\circ 21.8'$, $C = 140^\circ 7.0'$.
5. $70^\circ 8.8'$

§ 152; page 133.

3. $a = 37^\circ 7.2'$, $b = 41^\circ 1.6'$, $c = 49^\circ 28.2'$.
4. $a = 101^\circ 34.4'$, $b = 49^\circ 50.4'$, $c = 59^\circ 37.2'$.
5. $a = 125^\circ 16.2'$, $b = 151^\circ 37.4'$, $c = 75^\circ 55.4'$.
6. $126^\circ 43.4'$.

§ 153; pages 135, 136.

4. $B = 22^\circ 7.4'$, $C = 112^\circ 17.0'$, $c = 36^\circ 28.0'$.
 5. Impossible. 6. $C = 90^\circ$, $A = 138^\circ 31.6'$, $a = 146^\circ 41.9'$.
 7. $C = 154^\circ 7.7'$, $B = 60^\circ 45.8'$, $b = 63^\circ 52.2'$.
 8. $A = 36^\circ 18.2'$, $C = 160^\circ 52.8'$, $c = 148^\circ 58.4'$;
- or, $A = 143^\circ 41.8'$, $C = 38^\circ 23.0'$, $c = 77^\circ 39.2'$.

§ 154; page 137.

2. $b = 8^\circ 50.4'$, $c = 27^\circ 37.0'$, $C = 79^\circ 9.2'$.
 3. Impossible. 4. $c = 90^\circ$, $B = 8^\circ 14.0'$, $b = 18^\circ 41.2'$.
 5. $a = 155^\circ 56.8'$, $b = 138^\circ 33.6'$, $B = 100^\circ 0'$.
 6. $a = 63^\circ 55.7'$, $c = 156^\circ 5.8'$, $C = 154^\circ 55.0'$;
 or, $a = 116^\circ 4.3'$, $c = 72^\circ 43.8'$, $C = 87^\circ 24.0'$.

§ 155; pages 137, 138.

1. $A = 51^\circ 58.8'$, $B = 83^\circ 55.2'$, $C = 58^\circ 53.8'$.
 2. $b = 115^\circ 3.3'$, $a = 85^\circ 16.0'$, $A = 81^\circ 24.0'$.
 3. $a = 95^\circ 37.8'$, $b = 41^\circ 52.2'$, $C = 110^\circ 48.2'$.
 4. $C = 65^\circ 23.3'$, $A = 96^\circ 8.0'$, $a = 99^\circ 40.0'$.
 5. $a = 68^\circ 25.2'$, $b = 71^\circ 10.8'$, $c = 56^\circ 56.8'$.
 6. $c = 90^\circ$, $B = 63^\circ 43.4'$, $b = 66^\circ 26.8'$.
 7. $A = 121^\circ 32.8'$, $B = 40^\circ 56.8'$, $c = 37^\circ 25.8'$.
 8. Impossible.
 9. $A = 142^\circ 32.8'$, $B = 27^\circ 52.6'$, $C = 32^\circ 27.2'$.
 10. $b = 120^\circ 16.6'$, $c = 69^\circ 19.6'$, $A = 50^\circ 26.2'$.
 11. Impossible.
 12. $a = 100^\circ 8.4'$, $b = 50^\circ 1.8'$, $c = 60^\circ 6.0'$.
 13. $C = 90^\circ$, $B = 113^\circ 36.5'$, $b = 114^\circ 51.9'$.
 14. $a = 67^\circ 24.0'$, $C = 164^\circ 6.4'$, $c = 160^\circ 6.4'$;
 or, $a = 112^\circ 36.0'$, $C = 128^\circ 20.6'$, $c = 103^\circ 2.4'$.
 15. $C = 86^\circ 59.7'$, $A = 60^\circ 50.9'$, $b = 111^\circ 17.0'$.
 16. $C = 146^\circ 37.9'$, $B = 55^\circ 1.2'$, $b = 96^\circ 34.4'$.
 17. $a = 43^\circ 3.1'$, $b = 129^\circ 8.4'$, $B = 89^\circ 23.8'$;
 or, $a = 136^\circ 56.9'$, $b = 20^\circ 35.8'$, $B = 26^\circ 58.6'$.
 18. $A = 35^\circ 31.0'$, $B = 24^\circ 42.6'$, $C = 138^\circ 24.8'$.
 19. $B = 42^\circ 7.9'$, $C = 160^\circ 12.8'$, $c = 153^\circ 37.8'$;
 or, $B = 137^\circ 52.1'$, $C = 49^\circ 38.8'$, $c = 89^\circ 51.2'$.
 20. $a = 59^\circ 34.4'$, $b = 136^\circ 10.6'$, $c = 150^\circ 1.6'$.
 21. Impossible. 22. Impossible.
 23. $c = 64^\circ 19.4'$, $a = 34^\circ 3.0'$, $B = 37^\circ 39.6'$.
 24. $B = 68^\circ 18.0'$, $A = 132^\circ 33.8'$, $a = 131^\circ 15.8'$;
 or, $B = 111^\circ 42.0'$, $A = 77^\circ 4.6'$, $a = 95^\circ 50.0'$.
 25. $B = 134^\circ 57.3'$, $C = 50^\circ 41.1'$, $a = 69^\circ 8.8'$.
 26. $b = 27^\circ 22.1'$, $a = 117^\circ 9.2'$, $A = 47^\circ 21.2'$.

§ 157; page 140.

1. Bearing of Havana from Gibraltar, N. $77^{\circ} 40.3'$ W. ; of Gibraltar from Havana, N. $59^{\circ} 5.1'$ E. ; distance, 4593 mi.

2. Bearing of Batavia from San Francisco, N. $67^{\circ} 25.5'$ W. ; of San Francisco from Batavia, N. $47^{\circ} 12.3'$ E. ; distance, 8650 mi.

3. Bearing of Cape of Good Hope from Vera Cruz, S. $60^{\circ} 45.6'$ E. ; of Vera Cruz from Cape of Good Hope, N. $86^{\circ} 42.4'$ W. ; distance, 8330 mi.

4. Bearing of Callao from Auckland, S. $69^{\circ} 29.9'$ E. ; of Auckland from Callao, S. $50^{\circ} 2.5'$ W. ; distance, 6671 mi.

5. $54^{\circ} 35.9'$ N.

6. $51^{\circ} 48.0'$ S.

§ 160; pages 142, 143.

1. 3 h. 30 m. 40 s.

2. Hour of the day, 10 h. 23 m. 33.6 s., A.M. ; longitude, $19^{\circ} 6.6'$ W.

3. 7 h. 17 m. 14.4 s., P.M.

4. Hour of the day, 2 h. 45 m. 4 s., P.M. ; longitude, $66^{\circ} 59'$ W.

5. 6 h. 15 m. 21.6 s., A.M. 6. N. $80^{\circ} 24.6'$ W. 7. N. $71^{\circ} 43.7'$ E.

8. $45^{\circ} 5'$.

FOUR PLACE
LOGARITHMIC TABLES

TOGETHER WITH A

TABLE OF NATURAL SINES, COSINES,
TANGENTS, AND COTANGENTS

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D. C. HEATH & CO., PUBLISHERS

1908

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USE OF THE TABLES.

I. USE OF THE TABLE OF LOGARITHMS OF NUMBERS.

This table (pages 12 and 13) gives the mantissæ of the logarithms of all integers from 100 to 1000, calculated to four places of decimals.

TO FIND THE LOGARITHM OF A NUMBER OF THREE FIGURES.

Look in the column headed "No." for the first two significant figures of the given number.

Then the required mantissa will be found in the corresponding horizontal line, in the vertical column headed by the third figure of the number.

Finally, prefix the characteristic in accordance with the rules of §§ 66 or 67.

For example, $\log 168 = 2.2253$;
 $\log .0344 = 8.5366 - 10$; etc.

For a number consisting of one or two significant figures, the column headed 0 may be used.

Thus, let it be required to find $\log 83$ and $\log 9$.

By § 80, $\log 83$ has the same mantissa as $\log 830$, and $\log 9$ the same mantissa as $\log 900$.

Hence, $\log 83 = 1.9191$, and $\log 9 = 0.9542$.

TO FIND THE LOGARITHM OF A NUMBER OF MORE THAN THREE FIGURES.

Ex. 1. Required the logarithm of 327.6.

From the table, $\log 327 = 2.5145$,

and $\log 328 = 2.5159$.

That is, an increase of one unit in the number produces an increase of .0014 in the logarithm.

Therefore, an increase of .6 of a unit in the number will produce an increase of $.6 \times .0014$ in the logarithm, or .0008 to the nearest fourth decimal place.

$$\text{Hence, } \log 327.6 = 2.5145 + .0008 = 2.5153.$$

Note I. The above method is based on the assumption that the differences of logarithms are proportional to the differences of their corresponding numbers; which, though not strictly accurate, is sufficiently exact for practical purposes.

Note II. The difference between any mantissa in the table and the mantissa of the next higher number of three figures, is called the *tabular difference*. The subtraction may be performed mentally.

The following rule is derived from the above:

Find from the table the mantissa of the first three significant figures, and the tabular difference.

Multiply the latter by the remaining figures of the number with a decimal point before them. (See Note III.)

Add the result to the mantissa of the first three significant figures, and prefix the proper characteristic.

Note III. In finding the correction to the nearest unit's figure, the decimal portion should be omitted provided that, if it is .5 or more than .5, the unit's figure is increased by 1. Thus, 13.26 would be taken as 13; 30.5 as 31; 22.803 as 23.

Ex. 2. Find the logarithm of .021508.

Mantissa 215 = 3324	Tabular difference = 21
<u>2</u>	<u>.08</u>
3326	Correction = 1.68 = 2, nearly.
	Result, 8.3326 - 10.

TO FIND THE NUMBER CORRESPONDING TO A LOGARITHM.

Ex. 1. Required the number whose logarithm is 1.6571.
Find in the table the mantissa 6571.

In the corresponding line, in the column headed "No.," we find 45, the first two figures of the required number, and at the head of the column we find 4, the third figure.

Since the characteristic is 1, there must be two places to the left of the decimal point (§ 66).

Hence, the number corresponding to 1.6571 is 45.4.

Ex. 2. Required the number whose logarithm is 2.3934.

We find in the table the mantissæ 3927 and 3945, whose corresponding numbers are 247 and 248, respectively.

That is, an increase of 18 in the mantissa produces an increase of one unit in the number corresponding.

Therefore, an increase of 7 in the mantissa will produce an increase of $\frac{7}{18}$ of a unit in the number, or .4, nearly.

Hence, the number corresponding is $247 + .4$, or 247.4.

The following rule is derived from the above:

Find from the table the next less mantissa, the three figures corresponding, and the tabular difference.

Subtract the next less from the given mantissa, and divide the remainder by the tabular difference. (See Note V.)

Annex the quotient to the first three figures of the number, and point off the result. (See Note IV.)

Note IV. The rules for pointing off are the reverse of the rules for characteristic given in §§ 66 and 67.

1. *If -10 is not written after the mantissa, add 1 to the characteristic, giving the number of places to the left of the decimal point.*

2. *If -10 is written after the mantissa, subtract the positive part of the characteristic from 9, giving the number of ciphers to be placed between the decimal point and first significant figure.*

Ex. 3. Find the number whose logarithm is 8.5265 - 10.

5265

Next less mantissa = 5263 ; three figures corresponding, 336.

Tabular difference = 13)2.00(.15 = .2, nearly.

$$\begin{array}{r} 13 \\ \hline 70 \end{array}$$

By the rule of Note IV., there will be one cipher between the decimal point and first significant figure.

Hence, the number corresponding = .03362.

Note V. The correction can usually be depended upon to one decimal place; the division should be carried out to two decimal places in order to determine the last figure accurately. (See Note III.)

II. USE OF THE TABLE OF LOGARITHMIC SINE COSINES, ETC.

This table (pages 14 to 19) gives the logarithms of the sines, cosines, tangents, and cotangents of all angles at intervals of 10 minutes from 0° to 90° .

For angles between 0° and 45° , the degrees and minutes will be found in the *left-hand* column, and the functions in the columns designated by the names at the *top*; that is, sines in the first column, cosines in the second, tangents in the third, and cotangents in the fourth.

For angles between 45° and 90° , the degrees and minutes will be found in the *right-hand* column, and the functions in the columns designated by the names at the *foot*; that is, cosines in the first column, sines in the second, cotangents in the third, and tangents in the fourth.

If only the *mantissa* of the logarithm is found, the characteristic may be determined from the nearest logarithm in the same column in which the characteristic is given.

Since the sines and cosines of all acute angles, the tangents of angles between 0° and 45° , and the cotangents of angles between 45° and 90° , are less than unity, the characteristics of their logarithms have been increased by 10, and -10 must be written after the mantissa; in all other cases, the true value of the characteristic is given in the table.

Thus, $\log \sin 38^\circ 30' = 9.7941 - 10;$
 $\log \tan 65^\circ 20' = 0.3380;$
 $\log \cot 79^\circ 10' = 3.2819 - 10;$
 $\log \cos 89^\circ 40' = 7.7648 - 10; \text{ etc.}$

TO FIND THE LOGARITHMIC SINE, COSINE, TANGENT, OR
COTANGENT, OF ANY ACUTE ANGLE EXPRESSED
IN DEGREES AND MINUTES.

Find from the table the logarithmic sine, cosine, tangent, or cotangent of the next less multiple of ten minutes, and the difference for 1' corresponding. (See Note VI.)

Multiply this difference by the number of minutes remaining.

*If sine or tangent, add
If cosine or cotangent, subtract* } *this correction.*

Note VI. The columns immediately to the right of those headed "Sin.," "Cos.," and "Tan.," contain the respective differences for 1'; the right-hand column of differences is also to be used with the column headed "Cot."

It will be observed that the differences do not stand in the same horizontal line with the logarithms, but opposite the intervals between consecutive logarithms. For angles *below* 45°, the difference next *below* should be taken; for angles *above* 45°, the difference next *above*.

Note VII. The rule assumes that the differences of the logarithmic functions are proportional to the differences of their corresponding angles, which, unless the angle is near to 0° or 90°, is in general sufficiently exact for practical purposes. (See page 9.)

Note VIII. If the angle is expressed in degrees, minutes, and seconds, the seconds should be reduced to the decimal part of a minute before applying the rule.

Ex. 1. Find $\log \tan 17^\circ 14'$.

$\log \tan 17^\circ 10' = 9.4898 - 10$	D. 1' = 4.5
<div style="border-top: 1px solid black; margin-top: 5px;">18</div>	<div style="border-top: 1px solid black; margin-top: 5px;">4</div>
Result, $9.4916 - 10$	Corr. = 18.0

Ex. 2. Find $\log \cos 58^\circ 33.5'$.

$\log \cos 58^\circ 30' = 9.7181 - 10$	D. 1' = 2.1
<div style="border-top: 1px solid black; margin-top: 5px;">7</div>	<div style="border-top: 1px solid black; margin-top: 5px;">3.5</div>
Result, $9.7174 - 10$	<div style="border-top: 1px solid black; margin-top: 5px;">1 05</div>
	<div style="border-top: 1px solid black; margin-top: 5px;">6 3</div>
	7.35 = 7, nearly

TO FIND THE ACUTE ANGLE CORRESPONDING TO A GIVEN
LOGARITHMIC SINE, COSINE, TANGENT, OR COTANGENT.

Take from the table, if sine or tangent the next less, if cosine or cotangent the next greater, logarithmic function, the angle corresponding, and the difference for 1'. (See Note IX.)

Find the difference between the given logarithm and that taken from the table, and divide it by the difference for 1', giving the correction in minutes.

Add the result to the angle corresponding to the next less, or next greater, function.

Note IX. In searching for the next less (or greater) logarithm, attention must be paid to the fact that the functions are found in different columns according as the angle is below or above 45° .

If, for example, the next less logarithmic sine is found in the column with "Sin." at the *top*, the angle corresponding must be taken from the *left-hand* column; but if it is found in the column with "Sin." at the *foot*, the angle corresponding must be taken from the *right-hand* column. Similar considerations hold with respect to the other three functions.

Ex. 1. Find the angle whose $\log \sin = 9.9594 - 10$.

$$9.9594 - 10$$

Next less $\log \sin = 9.9590 - 10$; angle corresponding $= 65^\circ 30'$.

$$D. 1' = .6)4.0(6.66 = 6.7, \text{ nearly.}$$

Adding the correction, the result is $65^\circ 36.7'$.

Ex. 2. Find the angle whose $\log \cot = 0.1696$.

Next greater $\log \cot = 0.1710$; angle corresponding $= 34^\circ 0'$

$$\begin{array}{r} 0.1696 \\ \hline \end{array}$$

$$D. 1' = 2.7)14.0(5.18 = 5.2, \text{ nearly.}$$

$$\begin{array}{r} 13\ 5 \\ \hline 50 \\ 27 \\ \hline 230 \end{array}$$

Result, $34^\circ 5.2'$.

**TO FIND THE LOGARITHMIC SECANT OR COSECANT OF
ANY ACUTE ANGLE.**

Since $\sec x = \frac{1}{\cos x}$ and $\csc x = \frac{1}{\sin x}$, we have by § 85,

$$\log \sec x = \text{colog } \cos x, \text{ and } \log \csc x = \text{colog } \sin x.$$

Hence, *to find the logarithmic secant, subtract the logarithmic cosine from 10 - 10; and to find the logarithmic cosecant, subtract the logarithmic sine from 10 - 10.*

Ex. Find $\log \sec 22^\circ 38'$.

From the table, $\log \cos 22^\circ 38' = 9.9652 - 10$

Subtracting from 10 - 10, we have

$$\log \sec 22^\circ 38' = 0.0348.$$

Note X. The logarithmic cotangent of an angle may be obtained by subtracting the logarithmic tangent from 10 - 10.

**TO FIND THE LOGARITHMIC FUNCTIONS OF AN ANGLE NOT
LYING BETWEEN THE LIMITS 0° AND 90° .**

By § 34, any function of any angle may be expressed as a function of a certain acute angle; and hence the table of the functions of acute angles serves to determine the functions of angles of any magnitude whatever, positive or negative.

Ex. Find $\log \sin 152^\circ 16'$.

By § 34, $\sin 152^\circ 16' = \cos 62^\circ 16'$.

Then, $\log \sin 152^\circ 16' = \log \cos 62^\circ 16' = 9.6678 - 10$.

Another method would be to find the logarithmic sine of $27^\circ 44'$, the supplement of $152^\circ 16'$ (§ 32).

Note XI. If the natural function is *negative*, as for example in the case of the cosine of an angle between 90° and 180° , there is no logarithmic function, strictly speaking. (See Note before § 87.)

In solving examples involving such functions, we proceed as if the functions were positive, and determine the algebraic sign of the result irrespective of the logarithmic work. Illustrations of this will be found in Chapters X. and XI.

III. USE OF THE TABLE OF NATURAL SINES AND COSINES.

This table (pages 20 and 21) gives the natural values of the sines and cosines of all angles at intervals of 10 minutes from 0° to 90° , calculated to four places of decimals.

Its use is similar to that of the table of logarithmic functions, except that the tabular differences for $1'$ are not given, but are to be calculated from the table when required.

Ex. 1. Find $\sin 48^\circ 52'$.

The difference between $\sin 48^\circ 50'$ and $\sin 49^\circ 0'$ is .0019, one-tenth of which is .00019.

$$\sin 48^\circ 50' = .7528$$

$$D. 1' = 1.9$$

$$\begin{array}{r} 4 \\ \hline .7532, \text{ Ans.} \end{array}$$

$$\begin{array}{r} 2 \\ \hline \text{Corr.} = 3.8 = 4, \text{ nearly.} \end{array}$$

Ex. 2. Find the angle whose $\cos = .5506$.

The difference between the next greater and next less functions, .5519 and .5495, is .0024; one-tenth of which is .00024.

Next greater $\cos = .5519$; angle corresponding = $56^\circ 30'$.

$$\begin{array}{r} .5506 \\ D. 1' = 2.4 \end{array} 13.0 (5.41 = 5.4, \text{ nearly.}$$

$$\begin{array}{r} 12\ 0 \\ \hline 1\ 00 \\ 96 \\ \hline 40 \end{array}$$

Result, $56^\circ 35.4'$.

IV. USE OF THE TABLE OF NATURAL TANGENTS AND COTANGENTS.

This table (pages 22 and 23) gives the tangents and cotangents of all angles at intervals of 10 minutes from 0° to 90° ; its use is similar to that of the table of natural sines and cosines.

V. MORE ACCURATE METHOD FOR FINDING THE
LOGARITHMIC FUNCTIONS OF ANGLES NEAR
TO 0° OR 90° .

It was stated in Note VII., page 5, that in general the differences of the logarithmic functions are approximately proportional to the differences of their corresponding angles. It will be seen from the table that this is not the case with the logarithmic sines, tangents, and cotangents of angles near to 0° , nor with the logarithmic cosines, tangents, and cotangents of angles near to 90° .

Thus, the difference for $1'$ in the case of the logarithmic sine or tangent of an angle between $40'$ and $50'$ is 96.9, while for an angle between $50'$ and 1° it is 79.2.

A very accurate method for finding the logarithmic sine or tangent of an angle near to 0° , or the logarithmic cosine or cotangent of an angle near to 90° , is to first calculate the *natural function* by aid of the table of natural sines and cosines, or of natural tangents and cotangents, and then find the logarithm of the result.

To find the angle corresponding in similar cases, find the *number corresponding* to the logarithmic function, and then, by aid of the tables of natural functions, calculate the angle corresponding to the result.

Ex. 1. Find $\log \sin 0^\circ 56'$.

From the table of natural sines and cosines, we obtain

$$\text{natural } \sin 0^\circ 56' = .016289.$$

Whence, $\log \sin 0^\circ 56' = 8.2119 - 10.$

This result is correct to the last place of decimals; by the ordinary method we should have obtained $8.2102 - 10.$

Ex. 2. Find the angle whose $\log \tan = 8.0302 - 10.$

The number corresponding to this logarithm is .01072.

From the table of natural tangents and cotangents, the angle whose natural tangent is .01072 is $36.85'$.

This is correct to the last place of decimals ; the ordinary method would have given 37.15'.

Note XII. To find with accuracy the log cotangent of an angle near to 0° , find the log tangent of the angle by the above method, and then subtract the result from $10 - 10$. (See Note X., page 7.)

To find the angle corresponding to a log cotangent in a similar case, find the log tangent of the angle (Note X.), and then find the angle corresponding as above.

Note XIII. To find the log tangent of an angle near to 90° , find the log tangent of its *complement*, and subtract the result from $10 - 10$. (See Note XII.)

To find the angle corresponding in a similar case, find the angle corresponding to its cologarithm, and take the complement of the result.

Note XIV. The more accurate method should be employed in finding the log sines, tangents, or cotangents of angles between 0° and 5° , or the log cosines, tangents, or cotangents of angles between 85° and 90° , and the angles corresponding in similar cases. For angles between 5° and 85° the ordinary method is sufficiently exact.

FOUR PLACE LOGARITHMIC TABLES

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7350	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

Angle.	Sin.	D. 1'.	Cos.	D. 1'.	Tan.	D. 1'.	Cot.	
0° 0'	— ∞		10.0000		— ∞		∞	90° 0'
0° 10'	7.4637	301.1	.0000	.0	7.4637	301.1	2.5363	89° 50'
0° 20'	.7648	176.0	.0000	.0	.7648	176.1	.2352	89° 40'
0° 30'	.9408	125.0	.0000	.0	.9409	124.9	.0591	89° 30'
0° 40'	8.0658	96.9	.0000	.0	8.0658	96.9	1.9342	89° 20'
0° 50'	.1627	79.2	.0000	.1	.1627	79.2	.8373	89° 10'
1° 0'	8.2419	66.9	9.9999	.0	8.2419	67.0	1.7581	88° 0'
1° 10'	.3088	58.0	.9999	.0	.3089	58.0	.6911	88° 50'
1° 20'	.3668	51.1	.9999	.0	.3669	51.2	.6331	88° 40'
1° 30'	.4179	45.8	.9999	.1	.4181	45.7	.5819	88° 30'
1° 40'	.4637	41.3	.9998	.0	.4638	41.5	.5362	88° 20'
1° 50'	.5050	37.8	.9998	.1	.5053	37.8	.4947	88° 10'
2° 0'	8.5428	34.8	9.9997	.0	8.5431	34.8	1.4569	88° 0'
2° 10'	.5776	32.1	.9997	.1	.5779	32.2	.4221	87° 50'
2° 20'	.6097	30.0	.9996	.0	.6101	30.0	.3899	87° 40'
2° 30'	.6397	28.0	.9996	.1	.6401	28.1	.3599	87° 30'
2° 40'	.6677	26.3	.9995	.0	.6682	26.3	.3318	87° 20'
2° 50'	.6940	24.8	.9995	.1	.6945	24.9	.3055	87° 10'
3° 0'	8.7188	23.5	9.9994	.1	8.7194	23.5	1.2806	87° 0'
3° 10'	.7423	22.2	.9993	.0	.7429	22.3	.2571	86° 50'
3° 20'	.7645	21.2	.9993	.1	.7652	21.3	.2348	86° 40'
3° 30'	.7857	20.2	.9992	.1	.7865	20.2	.2135	86° 30'
3° 40'	.8059	19.2	.9991	.1	.8067	19.4	.1933	86° 20'
3° 50'	.8251	18.5	.9990	.1	.8261	18.5	.1739	86° 10'
4° 0'	8.8436	17.7	9.9989	.0	8.8446	17.8	1.1554	86° 0'
4° 10'	.8613	17.0	.9989	.1	.8624	17.1	.1376	85° 50'
4° 20'	.8783	16.3	.9988	.1	.8795	16.5	.1205	85° 40'
4° 30'	.8946	15.8	.9987	.1	.8960	15.8	.1040	85° 30'
4° 40'	.9104	15.2	.9986	.1	.9118	15.4	.0882	85° 20'
4° 50'	.9256	14.7	.9985	.2	.9272	14.8	.0728	85° 10'
5° 0'	8.9403	14.2	9.9983	.1	8.9420	14.3	1.0580	85° 0'
5° 10'	.9545	13.7	.9982	.1	.9563	13.8	.0437	84° 50'
5° 20'	.9682	13.4	.9981	.1	.9701	13.5	.0299	84° 40'
5° 30'	.9816	12.9	.9980	.1	.9836	13.0	.0164	84° 30'
5° 40'	.9945	12.5	.9979	.2	.9966	12.7	.0034	84° 20'
5° 50'	9.0070	12.2	.9977	.1	9.0093	12.3	0.9907	84° 10'
6° 0'	9.0192	11.9	9.9976	.1	9.0216	12.0	0.9784	84° 0'
6° 10'	.0311	11.5	.9975	.2	.0336	11.7	.9664	83° 50'
6° 20'	.0426	11.3	.9973	.1	.0453	11.4	.9547	83° 40'
6° 30'	.0539	10.9	.9972	.1	.0567	11.1	.9433	83° 30'
6° 40'	.0648	10.7	.9971	.2	.0678	10.8	.9322	83° 20'
6° 50'	.0755	10.4	.9969	.1	.0786	10.5	.9214	83° 10'
7° 0'	9.0859	10.2	9.9968	.2	9.0891	10.4	0.9109	83° 0'
7° 10'	.0961	9.9	.9966	.2	.0995	10.1	.9005	82° 50'
7° 20'	.1060	9.7	.9964	.1	.1096	9.8	.8904	82° 40'
7° 30'	.1157		.9963		.1194		.8806	82° 30'
	Cos.	D. 1'.	Sin.	D. 1'.	Cot.	D. 1'.	Tan.	Angle.

Angle.	Sin.	D. 1'.	Cos.	D. 1'.	Tan.	D. 1'.	Cot.	
7° 30'	.91157	9.5	.99963	.2	.91194	9.7	.08806	82° 30'
7° 40'	.1252	9.3	.99961	.2	.1291	9.4	.8709	82° 20'
7° 50'	.1345	9.1	.9959	.1	.1385	9.3	.8615	82° 10'
8° 0'	.91436	8.9	.99958	.2	.91478	9.1	.08522	82° 0'
8° 10'	.1525	8.7	.9956	.2	.1569	8.9	.8431	81° 50'
8° 20'	.1612	8.5	.9954	.2	.1658	8.7	.8342	81° 40'
8° 30'	.1697	8.4	.9952	.2	.1745	8.6	.8255	81° 30'
8° 40'	.1781	8.2	.9950	.2	.1831	8.4	.8169	81° 20'
8° 50'	.1863	8.0	.9948	.2	.1915	8.2	.8085	81° 10'
9° 0'	.91943	7.9	.99946	.2	.91997	8.1	.08003	81° 0'
9° 10'	.2022	7.8	.9944	.2	.2078	8.0	.7922	80° 50'
9° 20'	.2100	7.6	.9942	.2	.2158	7.8	.7842	80° 40'
9° 30'	.2176	7.5	.9940	.2	.2236	7.7	.7764	80° 30'
9° 40'	.2251	7.3	.9938	.2	.2313	7.6	.7687	80° 20'
9° 50'	.2324	7.3	.9936	.2	.2389	7.4	.7611	80° 10'
10° 0'	.92397	7.1	.99934	.3	.92463	7.3	.07537	80° 0'
10° 10'	.2468	7.0	.9931	.2	.2536	7.3	.7464	79° 50'
10° 20'	.2538	6.8	.9929	.2	.2609	7.1	.7391	79° 40'
10° 30'	.2606	6.8	.9927	.3	.2680	7.0	.7320	79° 30'
10° 40'	.2674	6.6	.9924	.2	.2750	6.9	.7250	79° 20'
10° 50'	.2740	6.6	.9922	.3	.2819	6.8	.7181	79° 10'
11° 0'	.92806	6.4	.99919	.2	.92887	6.6	.07113	79° 0'
11° 10'	.2870	6.4	.9917	.3	.2953	6.7	.7047	78° 50'
11° 20'	.2934	6.3	.9914	.2	.3020	6.5	.6980	78° 40'
11° 30'	.2997	6.1	.9912	.3	.3085	6.4	.6915	78° 30'
11° 40'	.3058	6.1	.9909	.2	.3149	6.3	.6851	78° 20'
11° 50'	.3119	6.0	.9907	.3	.3212	6.3	.6788	78° 10'
12° 0'	.93179	5.9	.99904	.3	.93275	6.1	.06725	78° 0'
12° 10'	.3238	5.8	.9901	.2	.3336	6.1	.6664	77° 50'
12° 20'	.3296	5.7	.9899	.3	.3397	6.1	.6603	77° 40'
12° 30'	.3353	5.7	.9896	.3	.3458	5.9	.6542	77° 30'
12° 40'	.3410	5.6	.9893	.3	.3517	5.9	.6483	77° 20'
12° 50'	.3466	5.5	.9890	.3	.3576	5.8	.6424	77° 10'
13° 0'	.93521	5.4	.99887	.3	.93634	5.7	.06366	77° 0'
13° 10'	.3575	5.4	.9884	.3	.3691	5.7	.6309	76° 50'
13° 20'	.3629	5.3	.9881	.3	.3748	5.6	.6252	76° 40'
13° 30'	.3682	5.2	.9878	.3	.3804	5.5	.6196	76° 30'
13° 40'	.3734	5.2	.9875	.3	.3859	5.5	.6141	76° 20'
13° 50'	.3786	5.1	.9872	.3	.3914	5.4	.6086	76° 10'
14° 0'	.93837	5.0	.99869	.3	.93968	5.3	.06032	76° 0'
14° 10'	.3887	5.0	.9866	.3	.4021	5.3	.5979	75° 50'
14° 20'	.3937	4.9	.9863	.4	.4074	5.3	.5926	75° 40'
14° 30'	.3986	4.9	.9859	.3	.4127	5.1	.5873	75° 30'
14° 40'	.4035	4.8	.9856	.3	.4178	5.2	.5822	75° 20'
14° 50'	.4083	4.7	.9853	.4	.4230	5.1	.5770	75° 10'
15° 0'	.94130		.99849		.94281		.05719	75° 0'
	Cos.	D. 1'.	Sin.	D. 1'.	Cot.	D. 1'.	Tan.	Angle.

Angle.	Sin.	D. 1'.	Cos.	D. 1'.	Tan.	D. 1'.	Cot.	
15° 0'	9.4130	4.7	9.9849	3	9.4281	5.0	0.5719	75° 0'
15° 10'	.4177	4.6	.9846	3	.4331	5.0	.5669	74° 50'
15° 20'	.4223	4.6	.9843	3	.4381	4.9	.5619	74° 40'
15° 30'	.4269	4.5	.9839	3	.4430	4.9	.5570	74° 30'
15° 40'	.4314	4.5	.9836	4	.4479	4.8	.5521	74° 20'
15° 50'	.4359	4.4	.9832	4	.4527	4.8	.5473	74° 10'
16° 0'	9.4403	4.4	9.9828	3	9.4575	4.7	0.5425	74° 0'
16° 10'	.4447	4.4	.9825	4	.4622	4.7	.5378	73° 50'
16° 20'	.4491	4.2	.9821	4	.4669	4.7	.5331	73° 40'
16° 30'	.4533	4.3	.9817	3	.4716	4.6	.5284	73° 30'
16° 40'	.4576	4.2	.9814	4	.4762	4.6	.5238	73° 20'
16° 50'	.4618	4.1	.9810	4	.4808	4.5	.5192	73° 10'
17° 0'	9.4659	4.1	9.9800	4	9.4853	4.5	0.5147	73° 0'
17° 10'	.4700	4.1	.9802	4	.4898	4.5	.5102	72° 50'
17° 20'	.4741	4.0	.9798	4	.4943	4.4	.5057	72° 40'
17° 30'	.4781	4.0	.9794	4	.4987	4.4	.5013	72° 30'
17° 40'	.4821	4.0	.9790	4	.5031	4.4	.4969	72° 20'
17° 50'	.4861	3.9	.9786	4	.5075	4.3	.4925	72° 10'
18° 0'	9.4900	3.9	9.9782	4	9.5118	4.3	0.4882	72° 0'
18° 10'	.4939	3.8	.9778	4	.5161	4.2	.4839	71° 50'
18° 20'	.4977	3.8	.9774	4	.5203	4.2	.4797	71° 40'
18° 30'	.5015	3.7	.9770	5	.5245	4.2	.4755	71° 30'
18° 40'	.5052	3.8	.9765	4	.5287	4.2	.4713	71° 20'
18° 50'	.5090	3.6	.9761	4	.5329	4.1	.4671	71° 10'
19° 0'	9.5126	3.7	9.9757	5	9.5370	4.1	0.4630	71° 0'
19° 10'	.5163	3.6	.9752	4	.5411	4.0	.4589	70° 50'
19° 20'	.5199	3.6	.9748	5	.5451	4.0	.4549	70° 40'
19° 30'	.5235	3.5	.9743	4	.5491	4.0	.4509	70° 30'
19° 40'	.5270	3.6	.9739	5	.5531	4.0	.4469	70° 20'
19° 50'	.5306	3.5	.9734	4	.5571	4.0	.4429	70° 10'
20° 0'	9.5341	3.4	9.9730	5	9.5611	3.9	0.4389	70° 0'
20° 10'	.5375	3.4	.9725	4	.5650	3.9	.4350	69° 50'
20° 20'	.5409	3.4	.9721	5	.5689	3.8	.4311	69° 40'
20° 30'	.5443	3.4	.9716	5	.5727	3.9	.4273	69° 30'
20° 40'	.5477	3.3	.9711	5	.5766	3.8	.4234	69° 20'
20° 50'	.5510	3.3	.9706	4	.5804	3.8	.4196	69° 10'
21° 0'	9.5543	3.3	9.9702	5	9.5842	3.7	0.4158	69° 0'
21° 10'	.5576	3.3	.9697	5	.5879	3.8	.4121	68° 50'
21° 20'	.5609	3.2	.9692	5	.5917	3.7	.4083	68° 40'
21° 30'	.5641	3.2	.9687	5	.5954	3.7	.4046	68° 30'
21° 40'	.5673	3.1	.9682	5	.5991	3.7	.4009	68° 20'
21° 50'	.5704	3.2	.9677	5	.6028	3.6	.3972	68° 10'
22° 0'	9.5736	3.1	9.9672	5	9.6064	3.6	0.3936	68° 0'
22° 10'	.5767	3.1	.9667	6	.6100	3.6	.3900	67° 50'
22° 20'	.5798	3.0	.9661	5	.6136	3.6	.3864	67° 40'
22° 30'	.5828		.9656		.6172		.3828	67° 30'
	Cos.	D. 1'.	Sin.	D. 1'.	Cot.	D. 1'.	Tan.	Angle.

Angle.	Sin.	D. 1'.	Cos.	D. 1'.	Tan.	D. 1'.	Cot.	
22° 30'	9.5828		9.9656		9.6172		0.3828	67° 30'
22° 40'	.5859	3.1	.9651	.5	.6208	3.6	.3792	67° 20'
22° 50'	.5880	3.0	.9646	.5	.6243	3.5	.3757	67° 10'
23° 0'	9.5919	3.0	9.9640	.6	9.6279	3.6	0.3721	67° 0'
23° 10'	.5948	2.9	.9635	.5	.6314	3.5	.3686	66° 50'
23° 20'	.5978	3.0	.9629	.6	.6348	3.4	.3652	66° 40'
23° 30'	.6007	2.9	.9624	.5	.6383	3.5	.3617	66° 30'
23° 40'	.6036	2.9	.9618	.6	.6417	3.4	.3583	66° 20'
23° 50'	.6065	2.9	.9613	.5	.6452	3.5	.3548	66° 10'
24° 0'	9.6093	2.8	9.9607	.6	9.6486	3.4	0.3514	66° 0'
24° 10'	.6121	2.8	.9602	.5	.6520	3.4	.3480	65° 50'
24° 20'	.6149	2.8	.9596	.6	.6553	3.3	.3447	65° 40'
24° 30'	.6177	2.8	.9590	.6	.6587	3.4	.3413	65° 30'
24° 40'	.6205	2.8	.9584	.6	.6620	3.3	.3380	65° 20'
24° 50'	.6232	2.7	.9579	.5	.6654	3.4	.3346	65° 10'
25° 0'	9.6259	2.7	9.9573	.6	9.6687	3.3	0.3313	65° 0'
25° 10'	.6286	2.7	.9567	.6	.6720	3.3	.3280	64° 50'
25° 20'	.6313	2.7	.9561	.6	.6752	3.2	.3248	64° 40'
25° 30'	.6340	2.7	.9555	.6	.6785	3.3	.3215	64° 30'
25° 40'	.6366	2.6	.9549	.6	.6817	3.2	.3183	64° 20'
25° 50'	.6392	2.6	.9543	.6	.6850	3.3	.3150	64° 10'
26° 0'	9.6418	2.6	9.9537	.6	9.6882	3.2	0.3118	64° 0'
26° 10'	.6444	2.6	.9530	.7	.6914	3.2	.3086	63° 50'
26° 20'	.6470	2.6	.9524	.6	.6946	3.2	.3054	63° 40'
26° 30'	.6495	2.5	.9518	.6	.6977	3.1	.3023	63° 30'
26° 40'	.6521	2.6	.9512	.6	.7009	3.2	.2991	63° 20'
26° 50'	.6546	2.5	.9505	.7	.7040	3.1	.2960	63° 10'
27° 0'	9.6570	2.4	9.9499	.6	9.7072	3.2	0.2928	63° 0'
27° 10'	.6595	2.5	.9492	.7	.7103	3.1	.2897	62° 50'
27° 20'	.6620	2.5	.9486	.6	.7134	3.1	.2866	62° 40'
27° 30'	.6644	2.4	.9479	.7	.7165	3.1	.2835	62° 30'
27° 40'	.6668	2.4	.9473	.6	.7196	3.1	.2804	62° 20'
27° 50'	.6692	2.4	.9466	.7	.7226	3.0	.2774	62° 10'
28° 0'	9.6716	2.4	9.9459	.7	9.7257	3.1	0.2743	62° 0'
28° 10'	.6740	2.4	.9453	.6	.7287	3.0	.2713	61° 50'
28° 20'	.6763	2.3	.9446	.7	.7317	3.0	.2683	61° 40'
28° 30'	.6787	2.4	.9439	.7	.7348	3.1	.2652	61° 30'
28° 40'	.6810	2.3	.9432	.7	.7378	3.0	.2622	61° 20'
28° 50'	.6833	2.3	.9425	.7	.7408	3.0	.2592	61° 10'
29° 0'	9.6856	2.3	9.9418	.7	9.7438	3.0	0.2562	61° 0'
29° 10'	.6878	2.2	.9411	.7	.7467	2.9	.2533	60° 50'
29° 20'	.6901	2.3	.9404	.7	.7497	3.0	.2503	60° 40'
29° 30'	.6923	2.2	.9397	.7	.7526	2.9	.2474	60° 30'
29° 40'	.6946	2.3	.9390	.7	.7556	3.0	.2444	60° 20'
29° 50'	.6968	2.2	.9383	.7	.7585	2.9	.2415	60° 10'
30° 0'	9.6990	2.2	9.9375	.8	9.7614	2.9	0.2386	60° 0'
	Cos.	D. 1'.	Sin.	D. 1'.	Cot.	D. 1'.	Tan.	Angle.

Angle.	Sin.	D. 1'.	Cos.	D. 1'.	Tan.	D. 1'.	Cot.	
30° 0'	9.6990	2.2	9.9375	.7	9.7614	3.0	0.2386	60° 0'
30° 10'	.7012	2.1	.9368	.7	.7644	2.9	.2356	59° 50'
30° 20'	.7033	2.2	.9361	.8	.7673	2.8	.2327	59° 40'
30° 30'	.7055	2.1	.9353	.7	.7701	2.9	.2299	59° 30'
30° 40'	.7076	2.1	.9346	.8	.7730	2.9	.2270	59° 20'
30° 50'	.7097	2.1	.9338	.7	.7759	2.9	.2241	59° 10'
31° 0'	9.7118	2.1	9.9331	.8	9.7788	2.8	0.2212	59° 0'
31° 10'	.7139	2.1	.9323	.8	.7816	2.9	.2184	58° 50'
31° 20'	.7160	2.1	.9315	.7	.7845	2.8	.2155	58° 40'
31° 30'	.7181	2.0	.9308	.8	.7873	2.9	.2127	58° 30'
31° 40'	.7201	2.1	.9300	.8	.7902	2.8	.2098	58° 20'
31° 50'	.7222	2.0	.9292	.8	.7930	2.8	.2070	58° 10'
32° 0'	9.7242	2.0	9.9284	.8	9.7958	2.8	0.2042	58° 0'
32° 10'	.7262	2.0	.9276	.8	.7986	2.8	.2014	57° 50'
32° 20'	.7282	2.0	.9268	.8	.8014	2.8	.1986	57° 40'
32° 30'	.7302	2.0	.9260	.8	.8042	2.8	.1958	57° 30'
32° 40'	.7322	2.0	.9252	.8	.8070	2.7	.1930	57° 20'
32° 50'	.7342	1.9	.9244	.8	.8097	2.8	.1903	57° 10'
33° 0'	9.7361	1.9	9.9236	.8	9.8125	2.8	0.1875	57° 0'
33° 10'	.7380	2.0	.9228	.9	.8153	2.7	.1847	56° 50'
33° 20'	.7400	1.9	.9219	.8	.8180	2.8	.1820	56° 40'
33° 30'	.7419	1.9	.9211	.8	.8208	2.7	.1792	56° 30'
33° 40'	.7438	1.9	.9203	.9	.8235	2.8	.1765	56° 20'
33° 50'	.7457	1.9	.9194	.8	.8263	2.7	.1737	56° 10'
34° 0'	9.7476	1.8	9.9186	.9	9.8290	2.7	0.1710	56° 0'
34° 10'	.7494	1.9	.9177	.8	.8317	2.7	.1683	55° 50'
34° 20'	.7513	1.8	.9169	.9	.8344	2.7	.1656	55° 40'
34° 30'	.7531	1.9	.9160	.9	.8371	2.7	.1629	55° 30'
34° 40'	.7550	1.8	.9151	.9	.8398	2.7	.1602	55° 20'
34° 50'	.7568	1.8	.9142	.9	.8425	2.7	.1575	55° 10'
35° 0'	9.7586	1.8	9.9134	.9	9.8452	2.7	0.1548	55° 0'
35° 10'	.7604	1.8	.9125	.9	.8479	2.7	.1521	54° 50'
35° 20'	.7622	1.8	.9116	.9	.8506	2.7	.1494	54° 40'
35° 30'	.7640	1.7	.9107	.9	.8533	2.6	.1467	54° 30'
35° 40'	.7657	1.8	.9098	.9	.8559	2.7	.1441	54° 20'
35° 50'	.7675	1.7	.9089	.9	.8586	2.7	.1414	54° 10'
36° 0'	9.7692	1.8	9.9080	1.0	9.8613	2.6	0.1387	54° 0'
36° 10'	.7710	1.7	.9070	.9	.8639	2.7	.1361	53° 50'
36° 20'	.7727	1.7	.9061	.9	.8666	2.6	.1334	53° 40'
36° 30'	.7744	1.7	.9052	.9	.8692	2.6	.1308	53° 30'
36° 40'	.7761	1.7	.9042	.9	.8718	2.7	.1282	53° 20'
36° 50'	.7778	1.7	.9033	1.0	.8745	2.6	.1255	53° 10'
37° 0'	9.7795	1.6	9.9023	.9	9.8771	2.6	0.1229	53° 0'
37° 10'	.7811	1.7	.9014	1.0	.8797	2.7	.1203	52° 50'
37° 20'	.7828	1.6	.9004	.9	.8824	2.6	.1176	52° 40'
37° 30'	.7844		.8995		.8850		.1150	52° 30'
	Cos.	D. 1'.	Sin.	D. 1'.	Cot.	D. 1'.	Tan.	Angle.

Angle.	Sin.	D. 1'.	Cos.	D. 1'.	Tan.	D. 1'.	Cot.	
37° 30'	.97844	1.7	.98995	1.0	.98850	2.6	.01150	52° 30'
37° 40'	.7801	1.6	.8985	1.0	.8876	2.6	.1124	52° 20'
37° 50'	.7877	1.6	.8975	1.0	.8902	2.6	.1098	52° 10'
38° 0'	.97893	1.7	.98965	1.0	.98928	2.6	.01072	52° 0'
38° 10'	.7910	1.6	.8955	1.0	.8954	2.6	.1046	51° 50'
38° 20'	.7926	1.6	.8945	1.0	.8980	2.6	.1020	51° 40'
38° 30'	.7941	1.5	.8935	1.0	.9006	2.6	.0994	51° 30'
38° 40'	.7957	1.6	.8925	1.0	.9032	2.6	.0968	51° 20'
38° 50'	.7973	1.6	.8915	1.0	.9058	2.6	.0942	51° 10'
39° 0'	.97989	1.5	.98905	1.0	.99084	2.6	.0916	51° 0'
39° 10'	.8004	1.6	.8895	1.1	.9110	2.5	.0890	50° 50'
39° 20'	.8020	1.6	.8884	1.0	.9135	2.6	.0865	50° 40'
39° 30'	.8035	1.5	.8874	1.0	.9161	2.6	.0839	50° 30'
39° 40'	.8050	1.6	.8864	1.1	.9187	2.5	.0813	50° 20'
39° 50'	.8066	1.5	.8853	1.0	.9212	2.6	.0788	50° 10'
40° 0'	.98081	1.5	.98843	1.1	.9238	2.6	.0762	50° 0'
40° 10'	.8096	1.5	.8832	1.1	.9264	2.5	.0736	49° 50'
40° 20'	.8111	1.4	.8821	1.1	.9289	2.6	.0711	49° 40'
40° 30'	.8125	1.5	.8810	1.0	.9315	2.6	.0685	49° 30'
40° 40'	.8140	1.5	.8800	1.1	.9341	2.5	.0659	49° 20'
40° 50'	.8155	1.4	.8789	1.1	.9366	2.6	.0634	49° 10'
41° 0'	.98109	1.5	.98778	1.1	.9392	2.5	.0608	49° 0'
41° 10'	.8184	1.4	.8767	1.1	.9417	2.6	.0583	48° 50'
41° 20'	.8198	1.5	.8756	1.1	.9443	2.5	.0557	48° 40'
41° 30'	.8213	1.4	.8745	1.2	.9468	2.6	.0532	48° 30'
41° 40'	.8227	1.4	.8733	1.1	.9494	2.5	.0506	48° 20'
41° 50'	.8241	1.4	.8722	1.1	.9519	2.5	.0481	48° 10'
42° 0'	.98255	1.4	.98711	1.2	.9544	2.6	.0456	48° 0'
42° 10'	.8269	1.4	.8699	1.1	.9570	2.5	.0430	47° 50'
42° 20'	.8283	1.4	.8688	1.2	.9595	2.6	.0405	47° 40'
42° 30'	.8297	1.4	.8676	1.1	.9621	2.5	.0379	47° 30'
42° 40'	.8311	1.3	.8665	1.2	.9646	2.5	.0354	47° 20'
42° 50'	.8324	1.4	.8653	1.2	.9671	2.6	.0329	47° 10'
43° 0'	.98338	1.3	.98641	1.2	.9697	2.5	.0303	47° 0'
43° 10'	.8351	1.4	.8629	1.1	.9722	2.5	.0278	46° 50'
43° 20'	.8365	1.3	.8618	1.2	.9747	2.5	.0253	46° 40'
43° 30'	.8378	1.3	.8606	1.2	.9772	2.6	.0228	46° 30'
43° 40'	.8391	1.4	.8594	1.2	.9798	2.5	.0202	46° 20'
43° 50'	.8405	1.3	.8582	1.3	.9823	2.5	.0177	46° 10'
44° 0'	.98418	1.3	.98509	1.2	.9848	2.6	.0152	46° 0'
44° 10'	.8431	1.3	.8557	1.2	.9874	2.5	.0126	45° 50'
44° 20'	.8444	1.3	.8545	1.3	.9899	2.5	.0101	45° 40'
44° 30'	.8457	1.2	.8532	1.2	.9924	2.5	.0076	45° 30'
44° 40'	.8469	1.3	.8520	1.3	.9949	2.6	.0051	45° 20'
44° 50'	.8482	1.3	.8507	1.2	.9975	2.5	.0025	45° 10'
45° 0'	.98495		.98495		.00000		.00000	45° 0'
	Cos.	D. 1'.	Sin.	D. 1'.	Cot.	D. 1'.	Tan.	Angle.

A.	Sin.	Cos.		A.	Sin.	Cos.		A.	Sin.	Cos.	
0°	.000000	1.0000	90°	30'	.1305	.9914	30'	15	.2588	.9659	75°
10'	.002909	1.0000	50'	40'	.1334	.9911	20'	10'	.2616	.9652	50'
20'	.005818	1.0000	40'	50'	.1363	.9907	10'	20'	.2644	.9644	40'
30'	.008727	1.0000	30'	8°	.1392	.9903	82°	30'	.2672	.9636	30'
40'	.011635	.9999	20'	10'	.1421	.9899	50'	40'	.2700	.9628	20'
50'	.014544	.9999	10'	20'	.1449	.9894	40'	50'	.2728	.9621	10'
1°	.017452	.9998	89°	30'	.1478	.9890	30'	16°	.2756	.9613	74°
10'	.02036	.9998	50'	40'	.1507	.9886	20'	10'	.2784	.9605	50'
20'	.02327	.9997	40'	50'	.1536	.9881	10'	20'	.2812	.9596	40'
30'	.02618	.9997	30'	9°	.1564	.9877	81°	30'	.2840	.9588	30'
40'	.02908	.9996	20'	10'	.1593	.9872	50'	40'	.2868	.9580	20'
50'	.03199	.9995	10'	20'	.1622	.9868	40'	50'	.2896	.9572	10'
2°	.03490	.9994	88°	30'	.1650	.9863	30'	17°	.2924	.9563	73°
10'	.03781	.9993	50'	40'	.1679	.9858	20'	10'	.2952	.9555	50'
20'	.04071	.9992	40'	50'	.1708	.9853	10'	20'	.2979	.9546	40'
30'	.04362	.9990	30'	10°	.1736	.9848	80°	30'	.3007	.9537	30'
40'	.04653	.9989	20'	10'	.1765	.9843	50'	40'	.3035	.9528	20'
50'	.04943	.9988	10'	20'	.1794	.9838	40'	50'	.3062	.9520	10'
3°	.05234	.9986	87°	30'	.1822	.9833	30'	18°	.3090	.9511	72°
10'	.05524	.9985	50'	40'	.1851	.9827	20'	10'	.3118	.9502	50'
20'	.05814	.9983	40'	50'	.1880	.9822	10'	20'	.3145	.9492	40'
30'	.06105	.9981	30'	11°	.1908	.9816	79°	30'	.3173	.9483	30'
40'	.06395	.9980	20'	10'	.1937	.9811	50'	40'	.3201	.9474	20'
50'	.06685	.9978	10'	20'	.1965	.9805	40'	50'	.3228	.9465	10'
4°	.06976	.9976	86°	30'	.1994	.9799	30'	19°	.3256	.9455	71°
10'	.07266	.9974	50'	40'	.2022	.9793	20'	10'	.3283	.9446	50'
20'	.07556	.9971	40'	50'	.2051	.9787	10'	20'	.3311	.9436	40'
30'	.07846	.9969	30'	12°	.2079	.9781	78°	30'	.3338	.9426	30'
40'	.08136	.9967	20'	10'	.2108	.9775	50'	40'	.3365	.9417	20'
50'	.08426	.9964	10'	20'	.2136	.9769	40'	50'	.3393	.9407	10'
5°	.08716	.9962	85°	30'	.2164	.9763	30'	20°	.3420	.9397	70°
10'	.09005	.9959	50'	40'	.2193	.9757	20'	10'	.3448	.9387	50'
20'	.09295	.9957	40'	50'	.2221	.9750	10'	20'	.3475	.9377	40'
30'	.09585	.9954	30'	13°	.2250	.9744	77°	30'	.3502	.9367	30'
40'	.09874	.9951	20'	10'	.2278	.9737	50'	40'	.3529	.9356	20'
50'	.10164	.9948	10'	20'	.2306	.9730	40'	50'	.3557	.9346	10'
6°	.10453	.9945	84°	30'	.2334	.9724	30'	21°	.3584	.9336	69°
10'	.10742	.9942	50'	40'	.2363	.9717	20'	10'	.3611	.9325	50'
20'	.11031	.9939	40'	50'	.2391	.9710	10'	20'	.3638	.9315	40'
30'	.11320	.9936	30'	14°	.2419	.9703	76°	30'	.3665	.9304	30'
40'	.11609	.9932	20'	10'	.2447	.9696	50'	40'	.3692	.9293	20'
50'	.11898	.9929	10'	20'	.2476	.9689	40'	50'	.3719	.9283	10'
7°	.12187	.9925	83°	30'	.2504	.9681	30'	22°	.3740	.9272	68°
10'	.12476	.9922	50'	40'	.2532	.9674	20'	10'	.3773	.9261	50'
20'	.12764	.9918	40'	50'	.2560	.9667	10'	20'	.3800	.9250	40'
30'	.13053	.9914	30'	15°	.2588	.9659	75°	30'	.3827	.9239	30'
	Cos.	Sin.	A.		Cos.	Sin.	A.		Cos.	Sin.	A.

A.	Sin.	Cos.		A.	Sin.	Cos.		A.	Sin.	Cos.	
30'	.3827	.9239	30'	30°	.5000	.8660	60°	30'	.6088	.7934	30'
40'	.3854	.9228	20'	10'	.5025	.8646	50'	40'	.6111	.7916	20'
50'	.3881	.9216	10'	20'	.5050	.8631	40'	50'	.6134	.7898	10'
23°	.3907	.9205	67°	30'	.5075	.8616	30'	38°	.6157	.7880	52°
10'	.3934	.9194	50'	40'	.5100	.8601	20'	10'	.6180	.7862	50'
20'	.3961	.9182	40'	50'	.5125	.8587	10'	20'	.6202	.7844	40'
30'	.3987	.9171	30'	31°	.5150	.8572	59°	30'	.6225	.7826	30'
40'	.4014	.9159	20'	10'	.5175	.8557	50'	40'	.6248	.7808	20'
50'	.4041	.9147	10'	20'	.5200	.8542	40'	50'	.6271	.7790	10'
24°	.4067	.9135	66°	30'	.5225	.8526	30'	39°	.6293	.7771	51°
10'	.4094	.9124	50'	40'	.5250	.8511	20'	10'	.6316	.7753	50'
20'	.4120	.9112	40'	50'	.5275	.8496	10'	20'	.6338	.7735	40'
30'	.4147	.9100	30'	32°	.5299	.8480	58°	30'	.6361	.7716	30'
40'	.4173	.9088	20'	10'	.5324	.8465	50'	40'	.6383	.7698	20'
50'	.4200	.9075	10'	20'	.5348	.8450	40'	50'	.6406	.7679	10'
25°	.4226	.9063	65°	30'	.5373	.8434	30'	40°	.6428	.7660	50°
10'	.4253	.9051	50'	40'	.5398	.8418	20'	10'	.6450	.7642	50'
20'	.4279	.9038	40'	50'	.5422	.8403	10'	20'	.6472	.7623	40'
30'	.4305	.9026	30'	33°	.5446	.8387	57°	30'	.6494	.7604	30'
40'	.4331	.9013	20'	10'	.5471	.8371	50'	40'	.6517	.7585	20'
50'	.4358	.9001	10'	20'	.5495	.8355	40'	50'	.6539	.7566	10'
26°	.4384	.8988	64°	30'	.5519	.8339	30'	41°	.6561	.7547	49°
10'	.4410	.8975	50'	40'	.5544	.8323	20'	10'	.6583	.7528	50'
20'	.4436	.8962	40'	50'	.5568	.8307	10'	20'	.6604	.7509	40'
30'	.4462	.8949	30'	34°	.5592	.8290	56°	30'	.6626	.7490	30'
40'	.4488	.8936	20'	10'	.5616	.8274	50'	40'	.6648	.7470	20'
50'	.4514	.8923	10'	20'	.5640	.8258	40'	50'	.6670	.7451	10'
27°	.4540	.8910	63°	30'	.5664	.8241	30'	42°	.6691	.7431	48°
10'	.4566	.8897	50'	40'	.5688	.8225	20'	10'	.6713	.7412	50'
20'	.4592	.8884	40'	50'	.5712	.8208	10'	20'	.6734	.7392	40'
30'	.4617	.8870	30'	35°	.5736	.8192	55°	30'	.6756	.7373	30'
40'	.4643	.8857	20'	10'	.5760	.8175	50'	40'	.6777	.7353	20'
50'	.4669	.8843	10'	20'	.5783	.8158	40'	50'	.6799	.7333	10'
28°	.4695	.8829	62°	30'	.5807	.8141	30'	43°	.6820	.7314	47°
10'	.4720	.8816	50'	40'	.5831	.8124	20'	10'	.6841	.7294	50'
20'	.4746	.8802	40'	50'	.5854	.8107	10'	20'	.6862	.7274	40'
30'	.4772	.8788	30'	36°	.5878	.8090	54°	30'	.6884	.7254	30'
40'	.4797	.8774	20'	10'	.5901	.8073	50'	40'	.6905	.7234	20'
50'	.4823	.8760	10'	20'	.5925	.8056	40'	50'	.6926	.7214	10'
29°	.4848	.8746	61°	30'	.5948	.8039	30'	44°	.6947	.7193	46°
10'	.4874	.8732	50'	40'	.5972	.8021	20'	10'	.6967	.7173	50'
20'	.4899	.8718	40'	50'	.5995	.8004	10'	20'	.6988	.7153	40'
30'	.4924	.8704	30'	37°	.6018	.7986	53°	30'	.7009	.7133	30'
40'	.4950	.8689	20'	10'	.6041	.7969	50'	40'	.7030	.7112	20'
50'	.4975	.8675	10'	20'	.6065	.7951	40'	50'	.7050	.7092	10'
30°	.5000	.8660	60°	30'	.6088	.7934	30'	45°	.7071	.7071	45°
	Cos.	Sin.	A.		Cos.	Sin.	A.		Cos.	Sin.	A.

A.	Tan.	Cot.		A.	Tan.	Cot.		A.	Tan.	Cot.	
0°	.000000	∞	90°	30'	.1317	7.5958	30'	15°	.2079	3.7321	75°
10'	.002909	343.7737	50'	40'	.1346	7.4287	20'	10'	.2711	3.6891	50'
20'	.005818	171.8854	40'	50'	.1376	7.2687	10'	20'	.2742	3.6470	40'
30'	.008727	114.5887	30'	8°	.1405	7.1154	82°	30'	.2773	3.6059	30'
40'	.011636	85.9398	20'	10'	.1435	6.9682	50'	40'	.2805	3.5656	20'
50'	.014545	68.7501	10'	20'	.1465	6.8269	40'	50'	.2836	3.5261	10'
1°	.017455	57.2900	89°	30'	.1495	6.6912	30'	16°	.2867	3.4874	74°
10'	.02036	49.1039	50'	40'	.1524	6.5606	20'	10'	.2899	3.4495	50'
20'	.02328	42.9641	40'	50'	.1554	6.4348	10'	20'	.2931	3.4124	40'
30'	.02619	38.1885	30'	9°	.1584	6.3138	81°	30'	.2962	3.3759	30'
40'	.02910	34.3678	20'	10'	.1614	6.1970	50'	40'	.2994	3.3402	20'
50'	.03201	31.2416	10'	20'	.1644	6.0844	40'	50'	.3026	3.3052	10'
2°	.03492	28.6363	88°	30'	.1673	5.9758	30'	17°	.3057	3.2709	73°
10'	.03783	26.4316	50'	40'	.1703	5.8708	20'	10'	.3089	3.2371	50'
20'	.04075	24.5418	40'	50'	.1733	5.7694	10'	20'	.3121	3.2041	40'
30'	.04366	22.9038	30'	10°	.1763	5.6713	80°	30'	.3153	3.1716	30'
40'	.04658	21.4704	20'	10'	.1793	5.5764	50'	40'	.3185	3.1397	20'
50'	.04949	20.2056	10'	20'	.1823	5.4845	40'	50'	.3217	3.1084	10'
3°	.05241	19.0811	87°	30'	.1853	5.3955	30'	18°	.3249	3.0777	72°
10'	.05533	18.0750	50'	40'	.1883	5.3093	20'	10'	.3281	3.0475	50'
20'	.05824	17.1693	40'	50'	.1914	5.2257	10'	20'	.3314	3.0178	40'
30'	.06116	16.3499	30'	11°	.1944	5.1446	79°	30'	.3346	2.9887	30'
40'	.06408	15.6048	20'	10'	.1974	5.0658	50'	40'	.3378	2.9600	20'
50'	.06700	14.9244	10'	20'	.2004	4.9894	40'	50'	.3411	2.9319	10'
4°	.06993	14.3007	86°	30'	.2035	4.9152	30'	19°	.3443	2.9042	71°
10'	.07285	13.7267	50'	40'	.2065	4.8430	20'	10'	.3476	2.8770	50'
20'	.07578	13.1969	40'	50'	.2095	4.7729	10'	20'	.3508	2.8502	40'
30'	.07870	12.7062	30'	12°	.2126	4.7041	78°	30'	.3541	2.8239	30'
40'	.08163	12.2505	20'	10'	.2156	4.6382	50'	40'	.3574	2.7980	20'
50'	.08456	11.8262	10'	20'	.2186	4.5730	40'	50'	.3607	2.7725	10'
5°	.08749	11.4301	85°	30'	.2217	4.5107	30'	20°	.3640	2.7475	70°
10'	.09042	11.0594	50'	40'	.2247	4.4494	20'	10'	.3673	2.7228	50'
20'	.09335	10.7119	40'	50'	.2278	4.3897	10'	20'	.3706	2.6985	40'
30'	.09629	10.3854	30'	13°	.2309	4.3315	77°	30'	.3739	2.6746	30'
40'	.09923	10.0780	20'	10'	.2339	4.2747	50'	40'	.3772	2.6511	20'
50'	.10216	9.7882	10'	20'	.2370	4.2193	40'	50'	.3805	2.6279	10'
6°	.10510	9.5144	84°	30'	.2401	4.1653	30'	21°	.3839	2.6051	69°
10'	.10805	9.2553	50'	40'	.2432	4.1126	20'	10'	.3872	2.5826	50'
20'	.11099	9.0098	40'	50'	.2462	4.0611	10'	20'	.3906	2.5605	40'
30'	.11394	8.7769	30'	14°	.2493	4.0108	76°	30'	.3939	2.5386	30'
40'	.11688	8.5555	20'	10'	.2524	3.9617	50'	40'	.3973	2.5172	20'
50'	.11983	8.3450	10'	20'	.2555	3.9136	40'	50'	.4006	2.4960	10'
7°	.12278	8.1443	83°	30'	.2586	3.8667	30'	22°	.4040	2.4751	68°
10'	.12574	7.9536	50'	40'	.2617	3.8208	20'	10'	.4074	2.4545	50'
20'	.12869	7.7704	40'	50'	.2648	3.7760	10'	20'	.4108	2.4342	40'
30'	.13165	7.5958	30'	15°	.2679	3.7321	75°	30'	.4142	2.4142	30'
	Cot.	Tan.	A.		Cot.	Tan.	A.		Cot.	Tan.	A.

A.	Tan.	Cot.		A.	Tan.	Cot.		A.	Tan.	Cot.	
30'	.4142	2.4142	30'	30°	.5774	1.7321	60°	30'	.7673	1.3032	30'
40'	.4176	2.3945	20'	10'	.5812	1.7205	50'	40'	.7720	1.2954	20'
50'	.4210	2.3750	10'	20'	.5851	1.7090	40'	50'	.7766	1.2876	10'
23°	.4245	2.3559	67°	30'	.5890	1.6977	30'	38°	.7813	1.2799	52°
10'	.4279	2.3369	50'	40'	.5930	1.6864	20'	10'	.7860	1.2723	50'
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24°	.4452	2.2460	66°	30'	.6128	1.6319	30'	39°	.8098	1.2349	51°
10'	.4487	2.2286	50'	40'	.6168	1.6212	20'	10'	.8146	1.2276	50'
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50'	.5735	1.7437	10'	20'	.7627	1.3111	40'	50'	.9942	1.0058	10'
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